Chapter 10

EXAMPLES OF CONTROL

This chapter is devoted to the development of examples of design. These examples at difference from the previous chapter miss any relationship with the classical approaches, and mostly cope with multivariable systems.
It is interesting to note that if from the one side familiar elements of the classical approach such as loop transfer function, sensitivity, etc. are not used directly, from the other the construction of the extended system with its weighting functions continues to represent a unifying point between the two approaches.
The first examples are a reformulation of the exercises of the course of Automatic Control of the colleague Prof. Giovanni Marro of the university of Bologna. The remaining have been suggested by the colleague Prof. Francesco Donati of Politecnico di Torino.

10.1 Control of the inverted pendulum

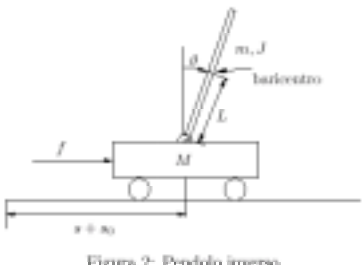

Fig. 10.1 Inverted pendulum

Equations of the model are

$$(M+m)\ddot{s} = mL\dot{\vartheta}^2 sin\,\vartheta - mL\ddot{\vartheta}\cos\vartheta + f$$
$$\left(J+mL^2\right)\ddot{\vartheta} = mgLsin\,\vartheta - mL\ddot{s}\cos\vartheta$$

We select as state variables

$$x_1 = s, x_2 = \dot{s}, x_3 = \vartheta, x_4 = \dot{\vartheta}$$

The dynamical behavior of the system is linearized in the neighbor of the value $\vartheta = 0$, so sinus is substituted with its argument and cosine with one.
Non-linearity is considered as multiplicative uncertainty where we assume the following range $\dot{\vartheta}^2 \le 0.025$. The linearized equations are

$$\begin{aligned}(M+m)\ddot{s} &= mLk(1+\Delta)\vartheta - mL\ddot{\vartheta} + f \\ (J+mL^2)\ddot{\vartheta} &= mgL\vartheta - mL\ddot{s}\end{aligned}$$

with the following parameters

$$M = 10Kg, m = 3Kg, L = 0.5m, J = m(2L)^2/3 = 3Kgm^2, g = 9.8m/\sec^2, k = 0.0125$$

The next figure shows the block diagram of the extended system. In the model the disturbance *w* has been added to the control, the non-linearity has been represented with the disturbance-objective pair $w_1 - z_1$, and a performance on the regulation error of s and ϑ due to *w* through the mask W_1:

$$A = \begin{bmatrix} -0.01 & 0 \\ 0 & -10 \end{bmatrix}, B = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}.$$

The available measures are s and ϑ.

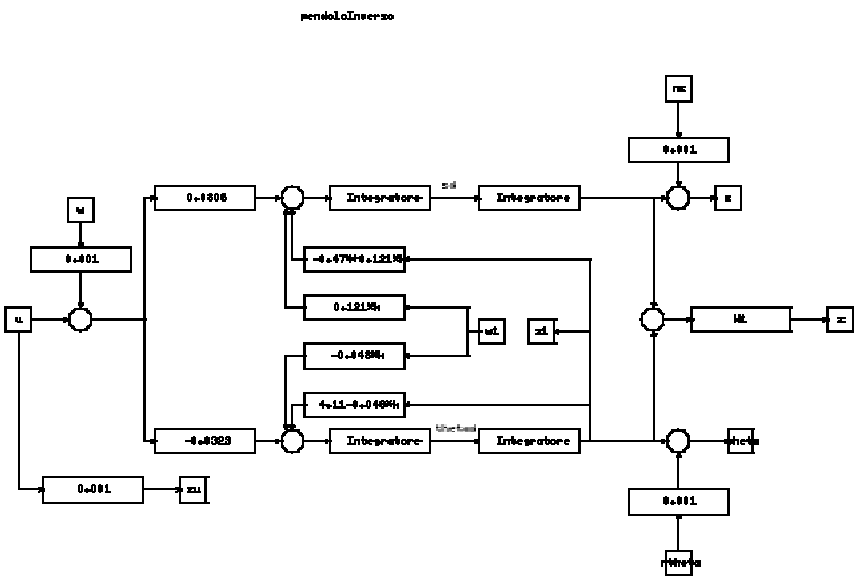

Fig. 10.2 Block diagram of the regulator of the inverted pendulum

The system is stabilisable and is detectable only measuring *s*. Nevertheless, performances are very poor if only one measure is used for the control.
The closed loop operator $w_1 - z_1$, with a H_∞ norm below –30 dB, shows that the system is strongly insensitive to non-linearity.
Let note that the property of reference tracking from *s* imposed by the weight W_1 in one degree of freedom design is accompanied by an unacceptable variation of the angle ϑ during the transients. For this, the previous scheme is transformed in a two degree of freedom design with the objective to track a reference of position of the chart. The new extended system is the following

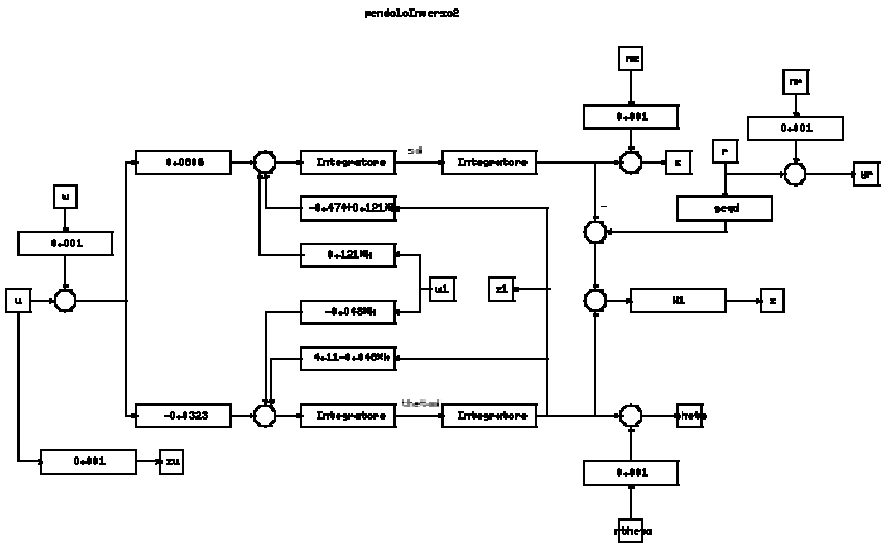

Fig. 10.3 Two degree of freedom design of the inverted pendulum

The new weighting function W_1 is now:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}.$$

The presence of a pole at the origin has the scope to guarantee a precise steady state of the closed loop function.

10.1.1 The equations of the dynamics

Dynamics equations of the inverted pendulum are derived as follows:
Co-ordinate of the center of weights of the pendulum

$$s + Lsin\vartheta, L\cos\vartheta.$$

Forces transferred from the pendulum to the joint

$$H(t) = m\frac{d^2}{dt^2}(s + Lsin\vartheta) = m\ddot{s} + mL\ddot{\vartheta}\cos\vartheta - mL\dot{\vartheta}^2 sin\vartheta$$

$$V(t) = mg + m\frac{d^2}{dt^2}(L\cos\vartheta) = mg - mL\ddot{\vartheta}sin\vartheta - mL\dot{\vartheta}^2\cos\vartheta$$

Equation of the dynamics of the chart

$$\begin{aligned} M\ddot{s} &= -H(t) + f \\ J\ddot{\vartheta} &= LV(t)sin\vartheta - LH(t)\cos\vartheta \end{aligned}.$$

10.2 Control of the Segway

Segway is an innovative personal transportation system for one passenger in equilibrium on a two wheeled chart. Motion is guaranteed by two electrical motors, one for each wheel, and equilibrium is maintained by the control system described in this section. This example, as the previous one, is characterizes by an inverse pendulum that can move longitudinally. In this case the joint of the pendulum coincides with the motor axis, and it transfers to the pendulum the motor torque.

Fig. 10.4 Segway

The model equations, transferring the wheel inertia directly in the chart mass, are

$$\begin{aligned} (M+m)\ddot{s} &= mL\dot{\vartheta}^2 sin\vartheta - mL\ddot{\vartheta}\cos\vartheta + C_m / r \\ \left(J + mL^2\right)\ddot{\vartheta} &= mgLsin\vartheta - mL\ddot{s}\cos\vartheta - C_m \end{aligned}$$

where $C_m = hu$ is the torque given by the motors and r is the ray of the wheels.
Let select the following state variables

$$x_1 = s, x_2 = \dot{s}, x_3 = \vartheta, x_4 = \dot{\vartheta}$$

We approximate the behavior of the system for small variations in the neighbor of the value $\vartheta = 0$, then we linearize sinus and cosine. The nonlinearity is treated as uncer-

tainty where we assume that $\dot{\vartheta}^2 \le 0.36$. We consider that the chart will subject to variable loads where the length of the center of weights L remains unchanged, but the mass m is variable. The linearized equations become

$$\left(M + \overline{m}\left(1 + k_m \Delta_m\right)\right)\ddot{s} = \overline{m}\left(1 + k_m \Delta_m\right)L\left(k\left(1 + \Delta\right)\vartheta - \ddot{\vartheta}\right) + C_m / r$$
$$\left(J + \overline{m}\left(1 + k_m \Delta_m\right)L^2\right)\ddot{\vartheta} = \overline{m}\left(1 + k_m \Delta_m\right)L\left(g\vartheta - \ddot{s}\right) - C_m$$

Realistic values of parameters of a full scale system are

$$M = 10Kg, m = 60 \div 130Kg, L = 1m, J = m\left(2L\right)^2 / 3 = 80 \div 173Kgm^2,$$
$$g = 9.8m/\sec^2, k = 0.18, r = 0.2, \overline{m} = 95, k_m = 0.268$$

Values of the parameters for a small scale model built using Lego Mindstorms are, instead

$$M + m = 0.4Kg, L = 0.1m, J + m \cdot L^2 \text{ (pendulum inertia at the joint)} = 1.55e-3Kgm^2,$$
$$g = 9.8m/\sec^2, k = 0.18, r = 0.04, k_m = 0.268, h = 1.2e-3$$

The block diagram of the extended system is presented in the next figure. An input disturbance w on the actuator has been added, The variability of the mass has been assumed as uncertainty represented by the four disturbance-objective pairs $w_1 : w_4 - z_1 : z_4$, a performance has bee imposed on the regulation errors of position s and angle ϑ using the weighting function W_1:

$$A = \begin{bmatrix} -1e-4 & 0 \\ 0 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We assume to measure s and ϑ.

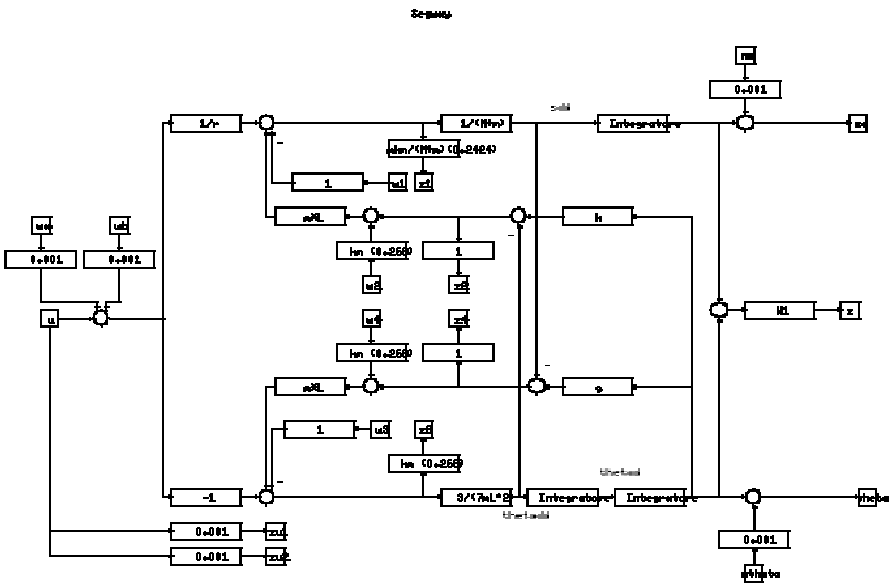

Fig. 10.5 Block diagram of the full scale Segway model

The whole design is centered on the robust stability at the change of load. It can be verified that if the coefficients of the weighting function of the figure are adopted the closed loop extended system H_∞ norm is 20 dB. Structuring the extended system the norm is –5 dB, this guaranteed robustness.
The coefficients of the diagonal of the structuring matrix D are:

$$D = diag\{0.05 \quad 2.1 \quad 0.07 \quad 1.8 \quad 0.004 \quad 0.065 \quad 1.52 \quad 1\}$$

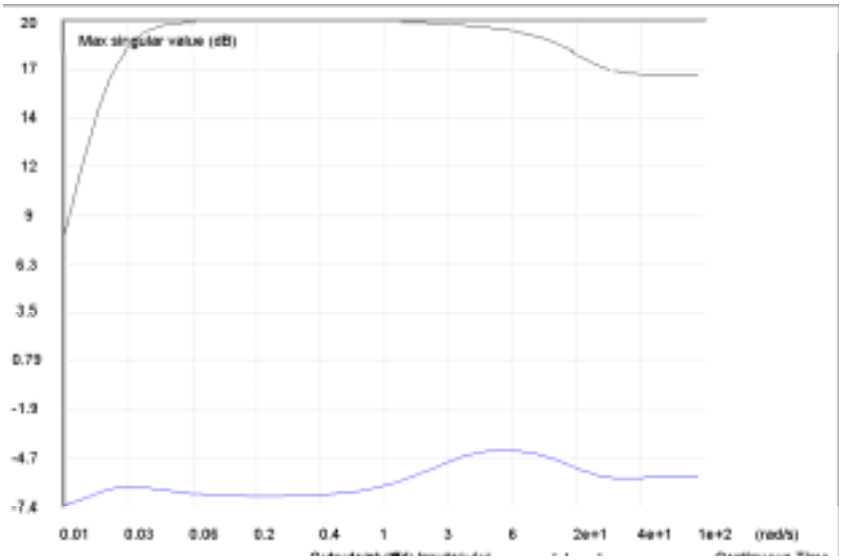

Fig. 10.6 Closed loop extended system without and with structuring

10.3 Non interactive control

This is a classical problem in multivariable controls. Let be a MIMO system with as many inputs as outputs interacting each other. We desire a closed loop control such that each output is controlled by just one reference independently from the others. The problem is cast in a two degree of freedom design where the multivariable Geq_{des} is constituted by a parallel of independent subsystems.
A two input- two output model is

$$A=\begin{bmatrix} -1 & 1 & 0 & 1 & 0 \\ 0 & -2.5 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix}, B=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$C=\begin{bmatrix} 1 & 2 & 0 & 0 & 0; \\ 2 & 1 & 0 & 4 & 1 \end{bmatrix}$$

The extended system built on the model is represented in the next figure

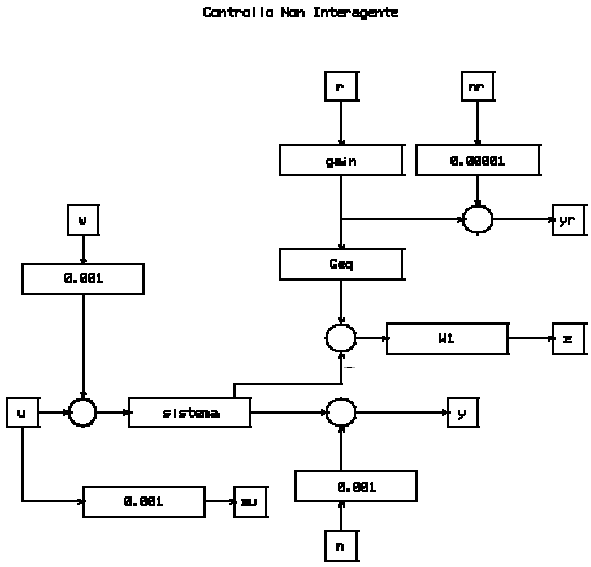

Fig. 10.7 Non-interacting control

A loop frequency band of about 1 rad/sec is required, assuming a weighting function W_1

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The presence of poles at the origin will guarantee the steady state gains of the closed loop. The selected Geq_{des} is given by the equations

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -36 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -225 & -15 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 36 & 0 \\ 0 & 0 \\ 0 & 225 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The four transfer functions in closed loop resulting from the design are shown in the next figure

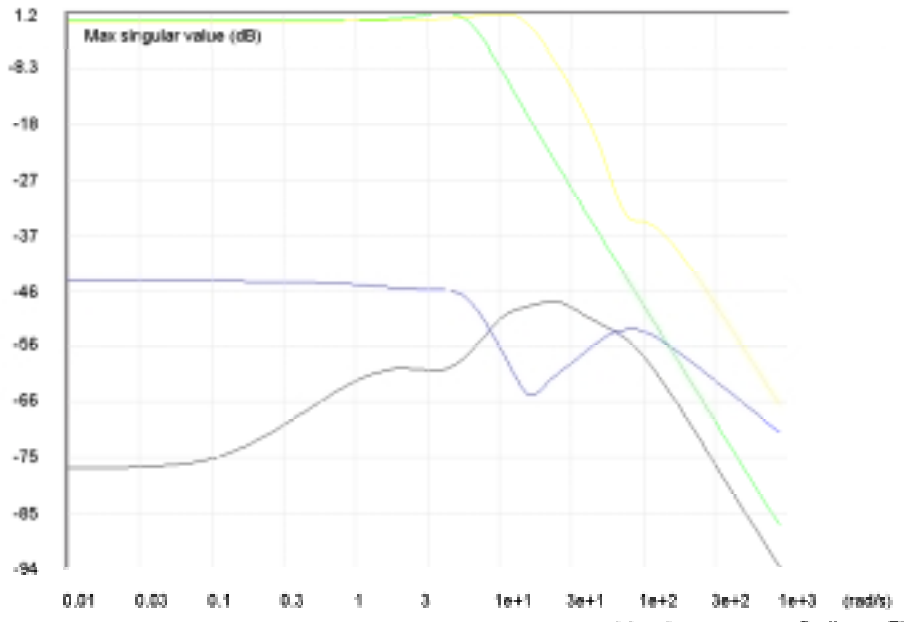

Fig. 10.8 Reference-output relationships in closed loop

The residual interaction between cross inputs and outputs has a gain below -45 dB.

10.4 Control of a hydroelectric plant

This section studies the control for opening and closing the valve of the turbine in a hydroelectric plant.
The characterizing aspect of this example is represented by the nonlinear dynamics that links gradient of pressure and flow in the model.

10.4.1 The model

The dynamics of the system shown in figure 10.9 is given by the following equations

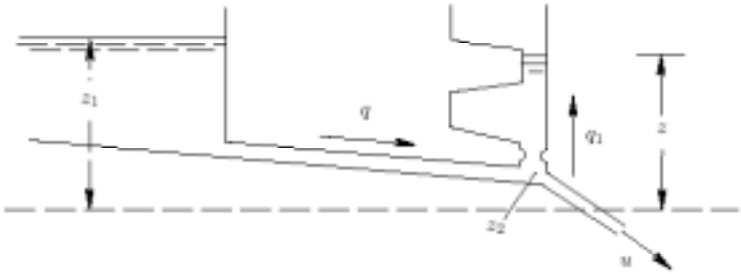

Fig. **10.9** The process

$$\begin{aligned} \dot{q}(t) &= -k_1|q(t)|q(t) + k_2\big(z_1(t) - z(t) - k_3|q(t)-u(t)|(q(t)-u(t))\big) \\ \dot{z}(t) &= F(z)\big(q(t)-u(t)\big) \end{aligned} \tag{10.22}$$

with outputs

$$\begin{aligned} &z(t) \\ &z_2(t) = z(t) + k_3|q(t)-u(t)|(q(t)-u(t)) \end{aligned}$$

where $q(t)$ is the flow in the forced conduct;
$z_1(t)$ is the water level in the dam;
$z(t)$ is the water level in the piezometric well;
$u(t);[0 \div 1]$ is the output flow to control;
$z_2(t)$ is the pressure expressed in level of water at the base of the piezometric well;
We assume the following values of the parameters

$$z_1 = 3; k_1 = 0.4; k_2 = 1; k_3 = 1; F(z) = 5\,.$$

We maneuver the output valve, opening it $u(t);[0 \to 1]$ with initial conditions $q(0) = 0; z(0) = 3$, or closing it $u(t);[1 \to 0]$ with initial conditions $q(0) = 1; z(0) = 2....$ (the final conditions resulting form the previous maneuver).
The dynamics of the system is poorly damped. We introduce a feedback control of the output flow $u(t)$ measuring outputs $z_2(t), z(t)$ to assist in the maneuver.
We first approach robust stability using a linear control substituting nonlinearities with multiplicative uncertainty.
Verified that in the transients

$$|q(t)| \le 1.4; |q(t)-u(t)| \le 1$$

equations (10.22) translate in

$$
\begin{aligned}
\dot{q}(t) &= -k_1 \cdot 0.7(1+\Delta_1)q(t) + k_2\left(z_1(t) - z_2(t)\right) \\
\dot{z}(t) &= F(z)\left(q(t) - u(t)\right) \\
z_2(t) &= z(t) + k_3 \cdot 0.5(1+\Delta_2)\left(q(t) - u(t)\right)
\end{aligned}
\tag{10.23}
$$

with $\|\Delta_1\|_\infty < 1, \|\Delta_2\|_\infty < 1$. The block diagram representing the model with multiplicative uncertainties is contained in figure 10.10

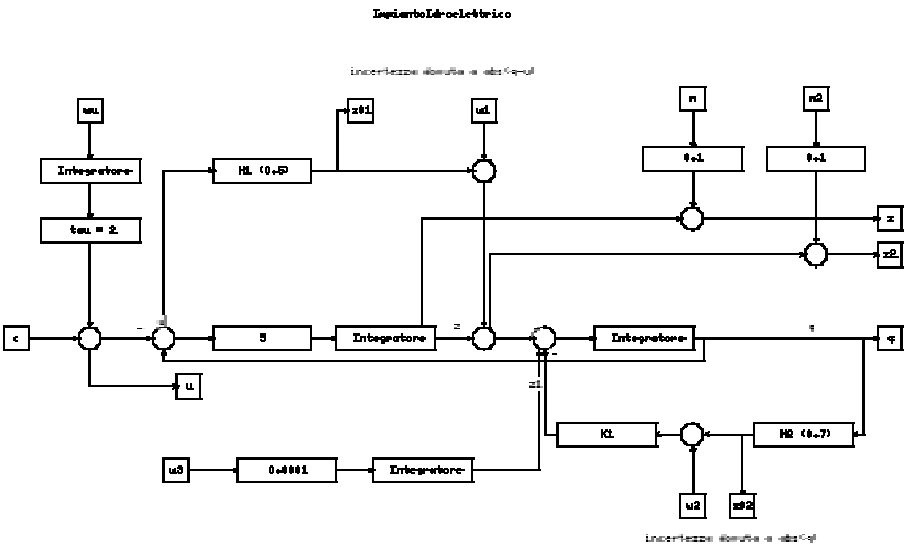

Fig. **10.10** Nominal model and uncertainties

Uncertainty on the water level in the dam is represented by an integrator with a small input noise, The action on the valve is composed of two additive terms: the open loop maneuver to open and close (0-1-0) plus a feedback controlled term *c*. The open loop control action is modeled as a disturbance and represented with an integrator followed by a time constant accounting for reasonable maneuvering times. Multiplicative uncertainties resulting from the linearizations are represented by two disturbance-objective pairs $w_1 - z_{01}, w_2 - z_{02}$.

10.4.2 Design

The available measures are z_2 and z with added measurement noises.

We close a control loop with constraints on the variations of z and on the control activity c.
Performance indices are assigned to z through the weighting function W_1 and to c through the weighting function W_u.
As the variations of z are of interest only during the transient, W_1 has a zero at the origin, viceversa, as the feedback control acts only during the transient W_u has a pole at the origin.

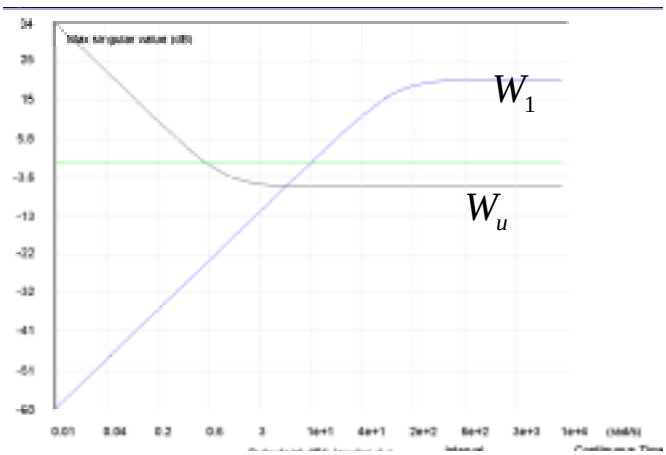

Fig. **10.11** The performance weighting functions

The resulting extended system is presented in the following figure

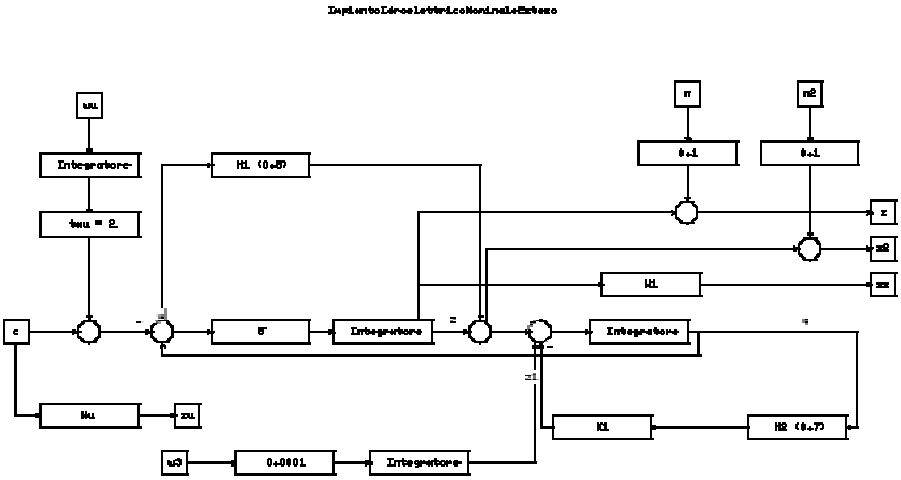

Fig. **10.12** The extended system representing nominal model and performances

The closed loop controlled system appears in figure 10.13.

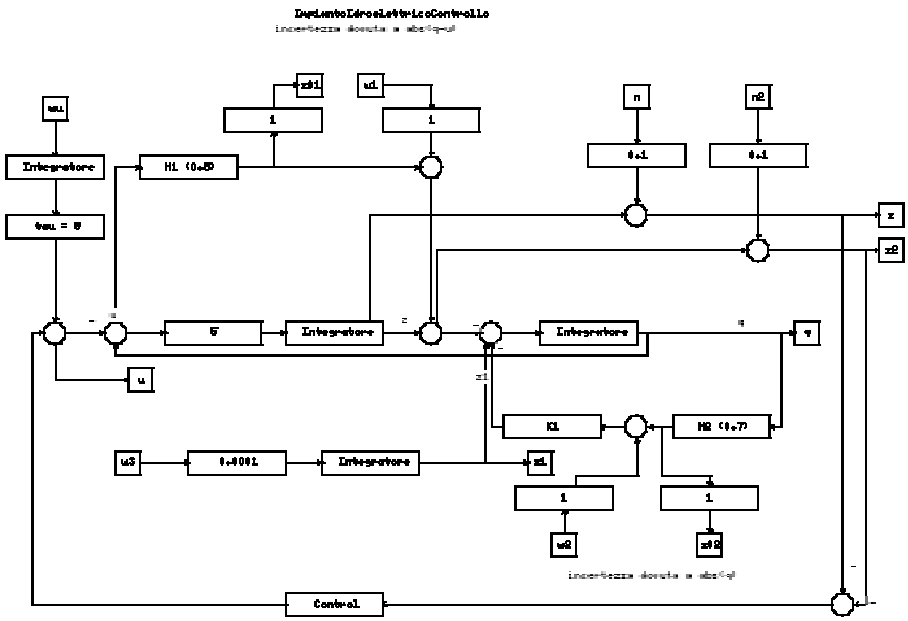

Fig. **10.13** Controlled system

Nominal responses of opening and closing maneuvers minimizing the norm H_2 of the extended system 10.12 with feedback from the outputs are in the next figure

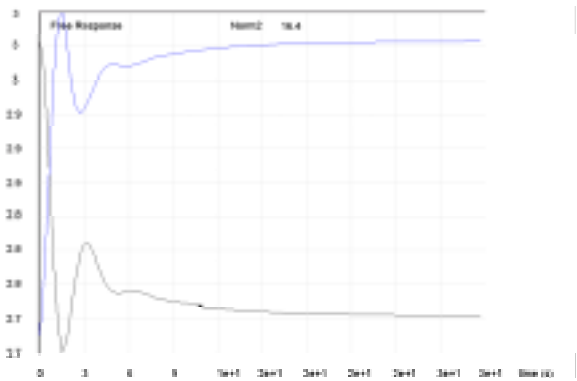

Fig. **10.14** Water level in the piezometric well

The result guarantees the robust stability, but is not insensitive to uncertainties. Responses for extreme values of Δ_1, Δ_2 are the following

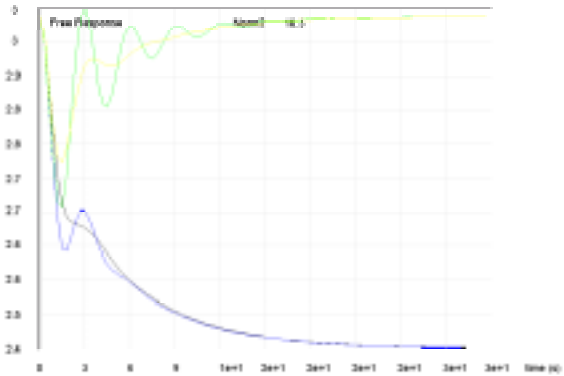

Fig. **10.15** Water level in the piezometric well in the presence of uncertainty

We modify the extended system accounting for uncertainties and performances, the design is performed minimizing the H_∞ norm.

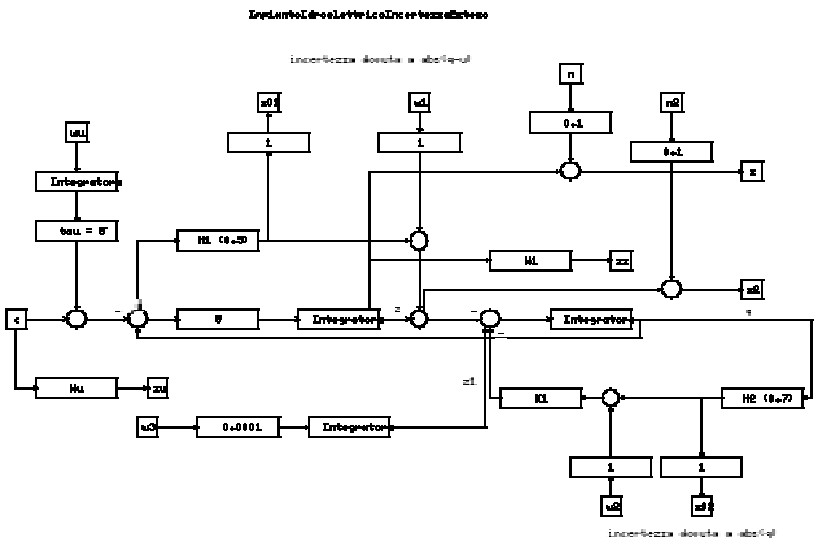

Fig. **10.16** Extended system encompassing performances and uncertainty

Let compare the closed loop operators $w_1, w_2 \rightarrow z_{01}, z_{02}$ showing robust stability indices of the two projects

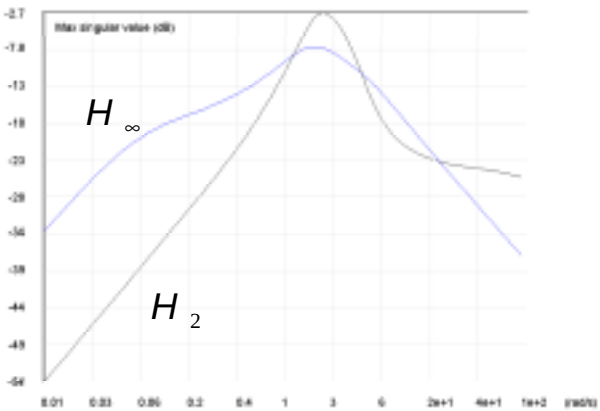

Fig. **10.17** Maximum singular values of the closed loop uncertainty operators

The next two graphics show the responses closing the valve in nominal conditions and for an extreme value of the uncertainty

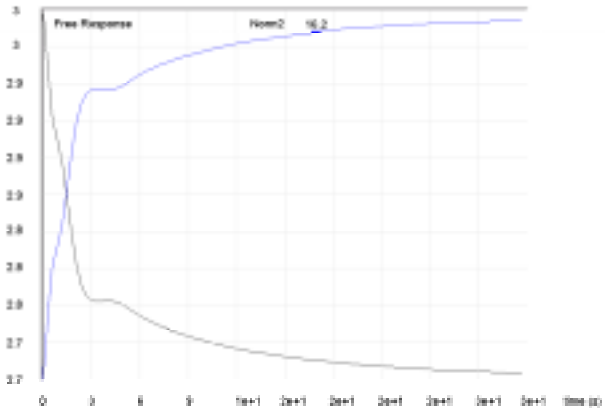

Fig. **10.18** Water level - closure – nominal and perturbed conditions

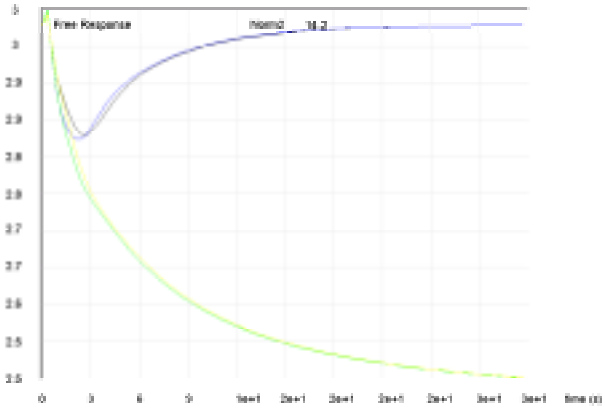

Fig. **10.19** Water level - opening – nominal and perturbed conditions

Let compare the previous results based on a linear model with the real response using the same control with the nonlinear model

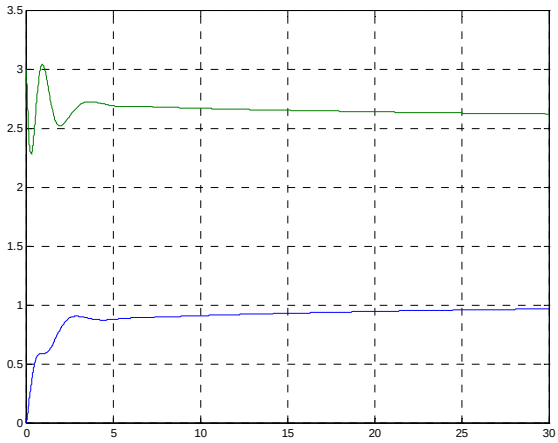

Fig. **10.20** Water level and flow opening maneuver – nonlinear system

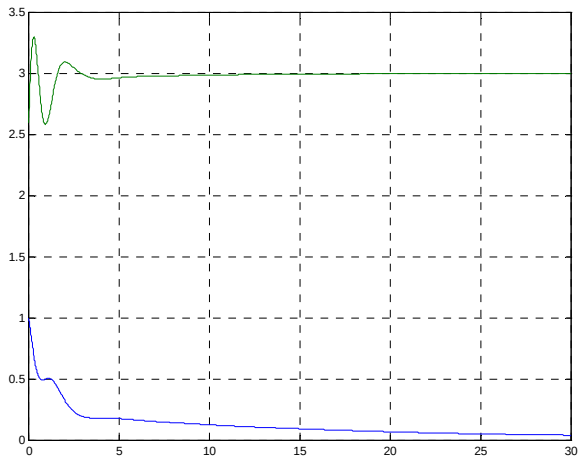

Fig. **10.21** Water level and flow closing maneuver – nonlinear system

10.4.3 Conclusions

In the examples of this section we have approached a strongly nonlinear plant modeling nonlinearity with multiplicative uncertainty. The maneuver of opening or closing the water valve has been modeled as an additive disturbance followed by an integrator. The

objective has been a compromise between control activity and variation of the water level in the piezometric well during the transient. The H_∞ norm of the closed loop uncertainty operator has been assumed as a measure of closed sensitivity to nonlinearity. The results show the superiority of robust control with respect to a control based only on the nominal model.

10.5 Two level position control

The problem is a precision longitudinal positioning with a two layers actuator based on electrical linear motors: the first layer to approach the target with a course precision, a lower frequency band and large excursion, the second layer, supported by the first one, for accurate tracking with greater band and accuracy, but limited excursion.

10.5.1 The model

The system is characterized by two slides driven by two linear electrical motors, the second slide mounted on top of the first one. The upper slide, with a smaller mass and reduced excursion supports directly the end effector with the objective to track fast variations of a reference target. The lower slide, supports the previous one is slower and maintains it near the target.

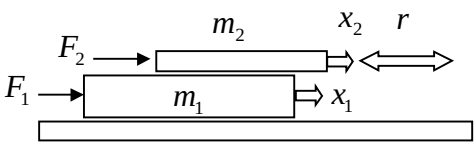

Fig. 10.22 Two layers mechanical positioning system

The equations of the behavior of the mechanical parts are the following:

$$m_1\ddot{x}_1 + \beta_2(\dot{x}_1 - \dot{x}_2) + \beta_1\dot{x}_1 = F_1 - F_2$$
$$m_2\ddot{x}_2 + \beta_2(\dot{x}_2 - \dot{x}_1) = F_2$$

with parameters

$$m_1 = 5kg,\ m_2 = 1kg,\ \beta_1 = \beta_2 = 0.001Ns/m$$

The electrical motors driving the masses have a variable gain factor modeled as multiplicative uncertainty

$$F_i = \tilde{k}_i \cdot V_i,\ i = 1,2,\ \tilde{k}_i = \bar{k}_i\left(1+\delta_i\Delta_{1i}\right),\ \left|\Delta_{1i}\right| < 1,\quad \bar{k}_i = 1,\ \delta_i = 0.5$$

parasitic dynamics at higher frequencies are present on each motor (servo1 e servo2)

$$G_{pi} = \frac{1}{s\tau_i + 1},\quad 0 < \tau_i < 0.01 \tag{10.24}$$

These dynamics will be represented as uncertainty

$$\tilde{G}_{pi} = \frac{200}{(s+200)\left(1+\dfrac{s}{s+200}\Delta_{2i}\right)},\quad \left|\Delta_{2i}\right| < 1 \tag{10.25}$$

The first mass is subject to an external force disturbance W_{11}, the two masses exchange a second force disturbance W_{12}. The extended system describing nominal model and uncertainty is described in the next block diagram

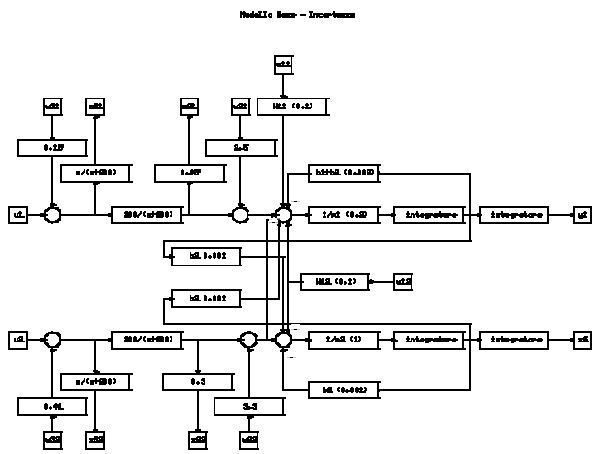

Fig. **10.23** Extended system of model and uncertainty

10.5.2 Requirements

The feedback control will be implemented measuring the position of the lower slide, and the relative position between the two slides.
Desired specifications require insensitivity of the end effector position from external disturbances and insensitivity of the closed loop function to uncertainty with significant attenuation (-20dB) up to 10 rad/s. The closed loop system response must have the approximate transfer function

$$G_{eqd}(s) = \frac{0.25}{s^2 + 0.5 \cdot s + 0.25} \cdot \tag{10.26}$$

Activities of the two controls and excursion of the signal $x_2 - x_1$, representing the relative position of the two slides, have to be limited.
As no precise requirements have been done, we have to preliminarily design an extended system and then to iterate on the weighting functions. We suggest the following steps.

Robust stability

Using the basic model with tentative weights on control and on output regulation we verify preliminarily robust stability. To do this we have conducted on this model a structured analysis on the uncertainty. As result we have established the coefficients that appear on the basic system 10.23.

Robustness and sensitivity

At the second step we assign sensitivity specification and we verify robust performance. We have adopted weighting function for sensitivity performances of the type

$$W_1(s) = \frac{1000}{s(s+10)} \tag{10.27}$$

adding fictitiously a disturbance on the output to weight sensitivity

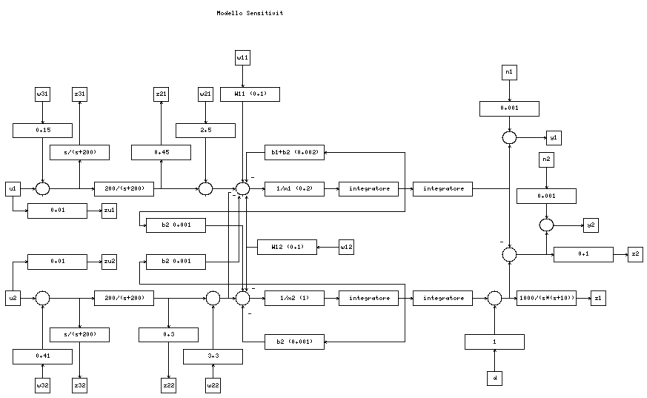

Fig. **10.24** Extended system with objective on the sensitivity

The best result carries to the following sensitivity function

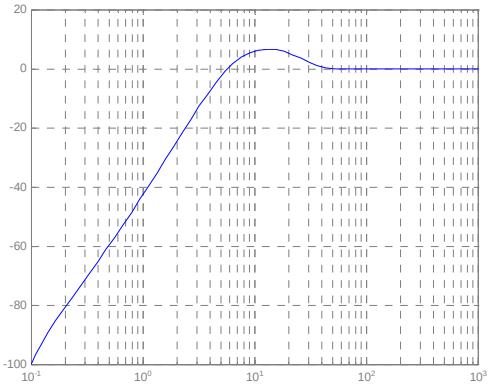

Fig. **10.25** Sensitivity function

Robust performance do not allow to reach the objective of 10 rad/s, as desired, but only 3 rad/s.

Reference tracking

The third step is devoted to reference tracking. We assign as weighting function of the tracking error the function $W_y(s) = \frac{1}{s}$. It has the purpose to guarantee steady state gain strictly equal to requirements, with a weighting gain reaching 1 at 1rad/sec (we expect to replicate the desired closed loop response up to 1rad/s).
Moreover, to penalize too large relative excursions between the two slides and maintain it near the center position at steady state, the signal $x_2 - x_1$ is added to the objectives with weight $W_{y_2}(s) = \frac{0.1}{s}$.
The revised extended system is the following

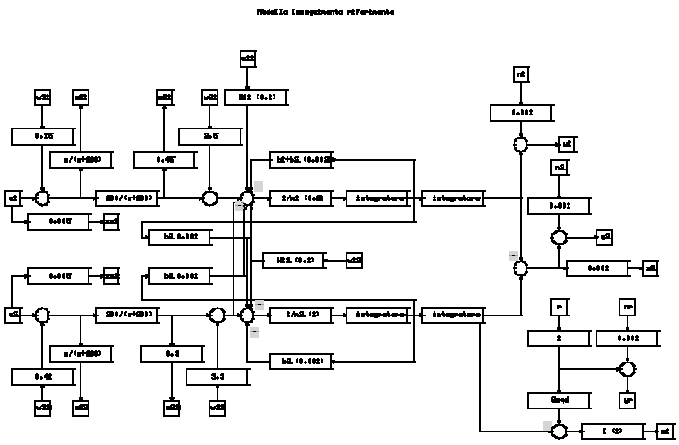

Fig. **10.26** Extended system with requirements on the tracking error

Responses resulting from a step of the reference are reported in the next figure

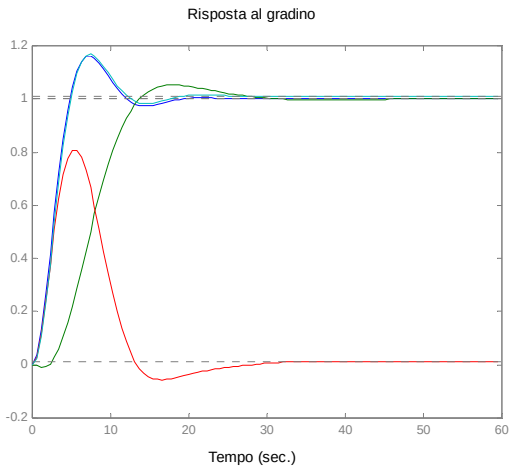

Fig. **10.27** Step response to a step – end effector, first and second slide

You will note that reference and output are very close, the slower slide reaches the steady state position, the faster response of the second slide contributes to the output in the transient, going to zero in the center of the first slide at steady state.

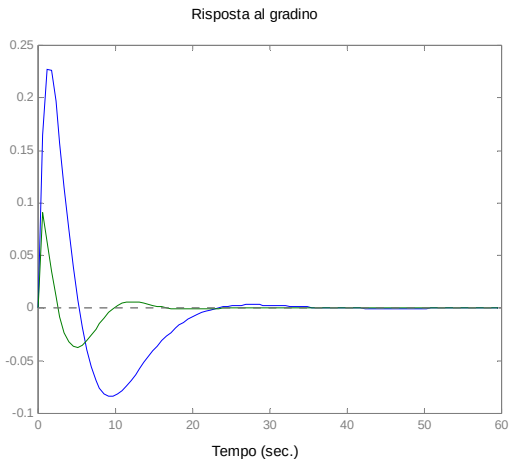

Fig. **10.28** Control activity of the two motors

During the previous step response are reported also the control activities of the two motors.
Sensitivity is, however, not completely satisfactory.

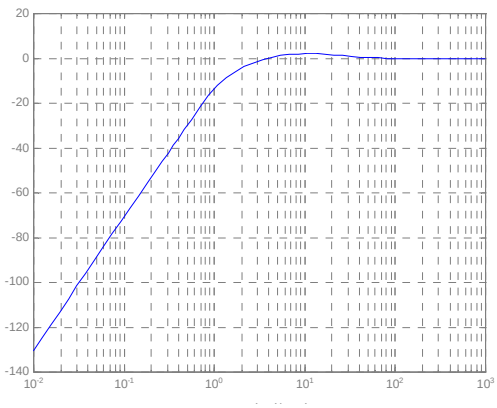

Fig. **10.29** Sensitivity

Digital control

To approach digital control we have sampled with a Z.O.H. the extended system 9.19. We tested a sampling period of 10 ms and 1 ms.

In both cases robust stability was reached marginally, as shown by the figure.

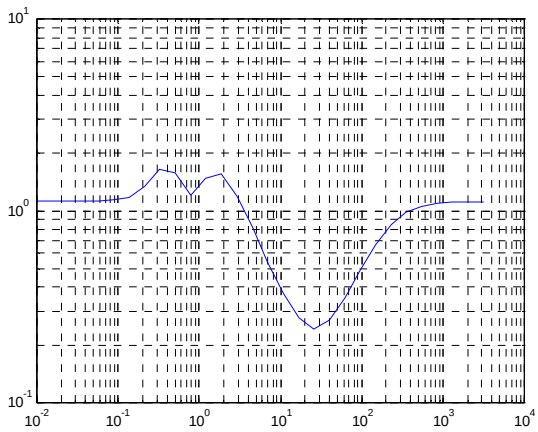

Fig. **10.30** Index of robust stability

Performances are, however, similar in the two cases. For a comparison with the analog system we consider here the design with sampling period of 1 ms.
The closed loop poles resulting from the synthesis in the continuous case are

$$
\begin{array}{cc}
0 & 0; \\
-45.2 & 45.2; \\
-45.2 & -45.2; \\
-200 & 0; \\
-200 & 0; \\
-14.3 & 14.3; \\
-14.3 & -14.3; \\
-24.7 & 0; \\
-200 & 0; \\
-200 & 0; \\
-13.3 & 0; \\
-1.81 & 0; \\
-0.842 & 1.16; \\
-0.842 & -1.16; \\
-0.447 & 0; \\
-0.202 & 0.268; \\
-0.202 & -0.268; \\
0 & 0; \\
-0.25 & 0.433; \\
-0.25 & -0.433; \\
-0.25 & 0.433; \\
-0.25 & -0.433
\end{array}
\tag{10.28}
$$

The closed loop poles in the discrete case are

real	imag	mod	sigma	omega
1	0	1	0	0;
0.955	0.0432	0.956	-45.2	45.2;
0.955	-0.0432	0.956	-45.2	-45.2;
0.819	0	0.819	-200	0;
0.986	0.0141	0.986	-14.3	14.3;
0.986	-0.0141	0.986	-14.3	-14.3;
0.819	0	0.819	-200	0;
0.819	0	0.819	-200	0;
0.981	0	0.981	-19.6	0;
0.819	0	0.819	-200	0;
0.99	0	0.99	-10.5	0;
0.998	0	0.998	-1.75	0;
0.999	0.00119	0.999	-0.832	1.19;
0.999	-0.00119	0.999	-0.832	-1.19;
1	0	1	-0.455	0;
1	0.000261	1	-0.217	0.261;
1	-0.000261	1	-0.217	-0.261;
1	0.000433	1	-0.25	0.433;
1	-0.000433	1	-0.25	-0.433;
1	0	1	0	0;
1	0.000433	1	-0.25	0.433;
1	-0.000433	1	-0.25	-0.433

(10.29)

Poles of the controller in the continuous case are

-43.9	45.8;
-43.9	-45.8;
-13.9	14.8;
-13.9	-14.8;
-30.8	0;
-15	0;
0	0;
0	0;
-0.25	0.433;
-0.25	-0.433

(10.30)

in the discrete case are

$$
\begin{array}{ccccc}
\text{real} & \text{imag} & \text{mod} & \text{sigma} & \text{omega} \\
0.956 & 0.0435 & 0.957 & -44.2 & 45.5; \\
0.956 & -0.0435 & 0.957 & -44.2 & -45.5; \\
0.986 & 0.0144 & 0.986 & -13.9 & 14.6; \\
0.986 & -0.0144 & 0.986 & -13.9 & -14.6; \\
0.975 & 0 & 0.975 & -25 & 0; \\
0.988 & 0 & 0.988 & -12.2 & 0; \\
1 & 0.000424 & 1 & -0.26 & 0.424; \\
1 & -0.000424 & 1 & -0.26 & -0.424; \\
1 & 0 & 1 & 0 & 0; \\
1 & 0 & 1 & 0 & 0
\end{array}
\tag{10.31}
$$