Appendix A2

GRAMIAN COMPUTATIONS

The design approach presented in this book is heavily dependent on gramian computations, i.e. the solution of Lyapunov equations both in the continuous and in the discrete case.
This issue is classical in the modern control theory. Nevertheless we found frequently numerical problems in the development of our algorithms using the various available techniques, especially when coping with non completely observable or controllable systems and singular gramian matrices [1]. Instead, the most robust approaches for solving the Lyapunov equations, that we experimented during our development and that finally adopted in our framework are described here. The first approach, we think is original, determines a cholesky factor of the gramian through orthogonal factorization. The second one use Linear Matrix Inequality (LMI).
Sincerely LMI appears the most powerful, coping easily with nearly singular problems and incorporating not only gramian computations but also H_∞ norm bound checking, and frequency constraints on the closed loop dynamics, however it sometime fails. So we resorted to integrate the two approaches in order to be able to reach always a solution.

A2.1 Gramian computation through orthogonal factorization

Let the Lyapunov equation with the gramian in a factorized form be

$$A'R'R + R'RA + D'D = 0 \tag{A2.1}$$

where R is a triangular factor of the gramian P, and D, as well, is a triangular factor of the constant matrix Q.
Transforms equation (A2.1) completing the square form in:

$$A'R'R + R'RA \pm (A'+I)R'R(A+I) + D'D = 0 \tag{A2.2}$$

that can be, also, written as

$$\frac{1}{2}(A'+I)R'R(A+I) + D'D = \frac{1}{2}(A'-I)R'R(A-I) \tag{A2.3}$$

Then, observe that, unless a left orthogonal transformation, the following two matrices are similar

$$\begin{bmatrix} \frac{1}{\sqrt{2}}R(A+I) \\ D \end{bmatrix} \propto \begin{bmatrix} \frac{1}{\sqrt{2}}R(A-I) \\ 0 \end{bmatrix} \tag{A2.4}$$

Based on (A2.4) the following recursive algorithm can be built on R, starting from an initial value R_0:

At iteration i, with R_i, performs the following triangular orthogonal factorization

$$Q\begin{bmatrix} \frac{1}{\sqrt{2}}R_i(A+I) \\ D \end{bmatrix} = \begin{bmatrix} H_i \\ 0 \end{bmatrix} \tag{A2.5}$$

were Q is orthogonal and H_i triangular, then compute

$$R_{i+1} = \sqrt{2}H_i(A-I)^{-1} \tag{A2.6}$$

and repeat until convergence.

Similarly, a still simpler algorithm can be used for the discrete time case.

$$R'R = A'R'RA + D'D \tag{A2.7}$$

$$\begin{bmatrix} RA \\ D \end{bmatrix} \propto \begin{bmatrix} R \\ 0 \end{bmatrix} \tag{A2.8}$$

So, the recursive formula is

$$Q\begin{bmatrix} R_iA \\ D \end{bmatrix} = \begin{bmatrix} R_{i+1} \\ 0 \end{bmatrix} \tag{A2.9}$$

A2.2 Gramian computations through LMI

The idea of using LMI for solving problems involving Lyapunov equations is based on the following property of monotonicity of the Lyapunov equation.

Property A2.1 *Monotonicity if the Lyapunov equation*

If there exists a solution of the Lyapunov equation

$$\begin{aligned} &A'X + XA + Q = 0 \\ &X \geq 0 \end{aligned} \tag{A2.10}$$

i.e. the matrix A represents an asymptotically stable system, then any symmetric positive semidefinite matrix P satisfying the inequality

$$\begin{aligned} &A'P + PA + Q \le 0 \\ &P \ge 0 \end{aligned} \tag{A2.11}$$

is an upper bound of X:

$$P \ge X \tag{A2.12}$$

In the literature are available numerical algorithms that solve linear programming problems where semidefinite positive symmetric matrices are involved (Semidefinite Programming), so the previous result can be used in different forms to evaluate the gramian, to find the optimum state feedback minimizing a square norm of the gramian, or guaranteeing an upper bound of the closed loop H_∞ operator norm.

In order to have a strong and reliable solution we resorted to use LMI and semidefinte programming with the following prototypical problem:

$$\begin{aligned} &\min x \quad \text{s.t.} \\ &A'P + PA + Q - Ix \le 0 \\ &P + Ix \ge 0 \\ &x \ge 0 \end{aligned} \tag{A2.13}$$

This problem has always a solution, however only if the solution is x=0, then P is an upper bound of the gramian X. Moreover the variable x is an invaluable indicator of the gap indicating how far we are from a solution.

The basic problem faced in solving observer or state feedback is minimizing the gramian of a closed loop system, let consider again the problem (3.34). The closed loop observability gramian (3.36) can be transformed as follows

$$W(A - BH)' + (A - BH)W + W(C - DH)'(C - DH)W = 0 \,. \tag{A2.14}$$

with $W = X^{-1}$. Then by introducing $Y = HW$:

$$WA' - Y'B' + AW - BY + (CW - DY)'(CW - DY) = 0 \,. \tag{A2.15}$$

This equation can be translated in the following LMI problem

$$\begin{aligned} &\min x \quad \text{s.t.} \\ &\left[\begin{array}{c|c} PA' - Y'B' + AP - BY & (CP - DY)' \\ \hline (CP - DY) & -I \end{array}\right] - Ix \le 0 \\ &P + Ix \ge 0 \\ &x \ge 0 \end{aligned} \tag{A2.16}$$

In a similar way the feedback that guarantees an upper bound of the H_∞ norm of the closed loop operator can be formulated as the following LMI

$$\min x \quad \text{s.t.}$$

$$\left[\begin{array}{cc|c} PA' - Y'B_2' + AP - B_2Y & B_1 & (C_1P - D_{12}Y)' \\ B_1' & -\gamma^2 I & D_{11}' \\ \hline (C_1P - D_{12}Y) & D_{11} & -I \end{array}\right] - Ix \le 0 \tag{A2.17}$$

$$P + Ix \ge 0$$

$$x \ge 0$$

In certain cases the requirements of the control problem are such that internal stability is not guarantee, or some of the closed loop poles are not moved enough inside the stability region. The LMI formulation allows to add constraints to regions of the achieved closed loop poles. We found especially useful to assign bounds to the minimum acceptable frequency for the closed loop poles. The inequality to be added to (A2.17) is the following:

$$P(A' + \alpha I) - Y'B_2' + (A + \alpha I)P - B_2Y - Ix \le 0 \tag{A2.18}$$

It ensures that the minimum real part of any closed loop pole is greater than α.

A2.3 Bibliography

[1] P.Hr. Petkov, N.D. Christov and M.M. Kostantinov, *Computational Methods for Linear Control Systems*. ISBN 0-13-161803-2 Prentice Hall New York (1991).