Chapter 6

CLASSICAL DESIGN

Algebraic techniques based on state variable representations and H_∞ analysis offer a very powerful tool, that will be shortly universally adopted for the synthesis of control loops, especially in the multivariable case. Numerical implementations, however, hide the basic principles that conduct to the results. In other words, the control works, but the designer doesn't understand how the result come from. Traditionally, viceversa, the design was much more transparent, carried on with graphical techniques based on the frequency analysis of the loop transfer function, and the designer always in full control of the situation and responsible for all choices. As a counterpart, classical techniques lack a real design algorithm, the approach asks for creativity from the designer, and most importantly, it applies to SISO systems. As soon as the system complexity grows, or from SISO we move to MIMO systems, the design becomes very involved.
This chapter offers a bridge between classical and modern approaches, but overall it explains in simple terms why a feedback control behaves as we expect. Starting from simple SISO cases it will become easy to understand results obtained with synthesis techniques in the much more complex situations of the next chapter.

6.1 Frequency analysis

Frequency analysis, that dominated the design of the control loop for many decades, has been revitalized from the H_∞ approach. In spite of the fact the controller in the synthesis is the result of algebraic techniques, frequency analysis enters in the treatment to represent the H_∞ norms of the closed loop operators subject to specifications. As we know this norm is the maximum absolute value or the maximum singular value, in the matrix case, of the transfer function in $j\omega$ of the closed input-output operators present in the loop. Closed loop transfer functions, in turn, can be related with open loop transfer functions (and easily visualized, at least in the simple cases) transferring to the latter the analysis problems.
Frequency analysis of the open loop transfer function, is in fact, the key point of the techniques that dominated the study of control systems for more than fifty years. The design is the result of shaping with a proper filter, called a compensation filter, in cascade to the system the behavior in frequency of the loop transfer function. For this reason these techniques are universally known in the literature as "*loop shaping*" techniques.

The tools used for the graphical representation in the frequency of the transfer function are , the Nyquist plot, the Bode diagram and the Nichols charts. A classical reference that started these approaches is [1]. With these tools the whole cycle from the analysis of stability, the design of the control, up the to the verification of requirements SISO can be conducted. This will be the subject of this chapter.
When dealing with sampled data systems, how we already sought in the introductory chapter, the *Z* transform of the transfer function computed for values of *z* on the unitary circle carries to a frequency representation, i.e.:

$$\hat{G}_a(s) = G_a(z)\big|_{z=e^{s\cdot\Delta}}$$

with this representation results are derived as for the continuous case, with the exception that the compensator will be implemented as a digital filter.
A few classical texts of analysis and design based on frequency techniques are [2], [3], [4]; recent new editions of classical texts are [3] è [5].

6.1.1 The loop transfer function

Let consider the transfer function $G_a(s)$ of a strictly proper SISO system. The most natural way to visualize this transfer function is to plot its values (real part on the abscissa and imaginary part on the ordinate) for each value of the frequency ω, i.e. when the argument *s* assumes values on the imaginary axis. This representation is called *Nyquist diagram*, or also *polar diagram* .

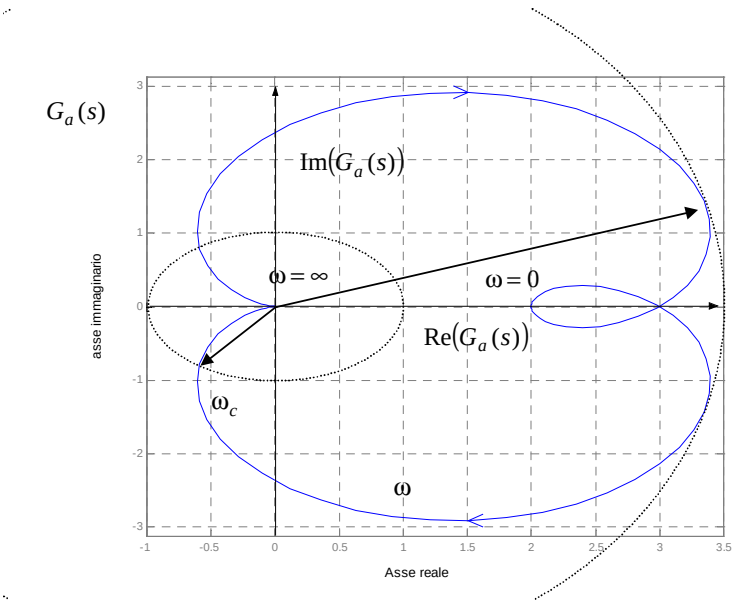


Fig. 6.1 Nyquist diagram

From this diagram, in particular, a few significant elements of the transfer function are immediately evident: the system steady state gain (the value of the function when $j\omega = 0$), when the system is stable the H_∞ norm (the maximum modulus), The value of the frequency ω_c where the modulus is one (The point where the polar plot crosses the unitary circle).
Alternatively, module and phase of the vector can be plotted separately on two distinct graphs, with the frequency on the abscissa. Customarily these plots use logarithmic scale for the abscissa and for the ordinate of the modulus. This representation is called *Bode diagram*.

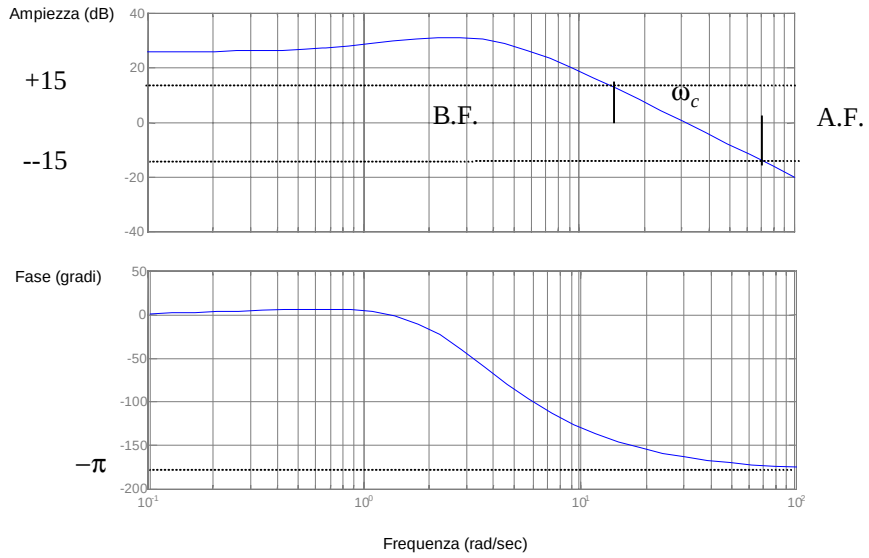


Fig. 6.2 Modulus and phase of a transfer function in a Bode diagram

Again, steady state gain, maximum gain, frequency when the gain is one are easily visualized. The analysis of the modulus of the loop transfer function of a control system is particularly interesting. Its behavior is generally similar to the one of figure 6.2. Three bands of frequency can be detected: a low frequency band, starting form frequency zero, where the gain of the transfer function is significantly greater than one, an intermediary band where the gain is approximately one, and a last high frequency band where the gain is much smaller than one[10].

[10] Conventionally values of +15dB and –15dB are assumed to discriminate high and low gains, respectively.

Two of the most significant transfer functions characterizing the closed loop, we outlined in **(4.11)** and **(4.12)**, are $G_{eq1}(s)=\dfrac{G_a(s)}{1+G_a(s)}$ and $S(s)=\dfrac{1}{1+G_a(s)}$, the closed loop transfer function with unitary feedback and the sensitivity function. They have the property $G_{eq1}(s)+S(s)=1$. As both functions depend exclusively on $G_a(s)$, it is easy to deduce their frequency behavior from $|G_a(s)|$.

Using the following simple approximations of the loop return transfer function

$$1+G_a(s)\approx\begin{cases}|G_a|<<1\Rightarrow 1\\ |G_a|>>1\Rightarrow G_a(s)\end{cases}, \tag{6.1}$$

we note that the two closed loop functions are very well characterized at least at low and at high frequencies by

$$G_{eq1}(s)\approx\begin{cases}1,\ BF\\ G_a(s),\ AF\end{cases} \tag{6.1}$$

$$S(s)\approx\begin{cases}G_a^{-1}(s),\ BF\\ 1,\ AF\end{cases} \tag{6.1}$$

as the next figure shows

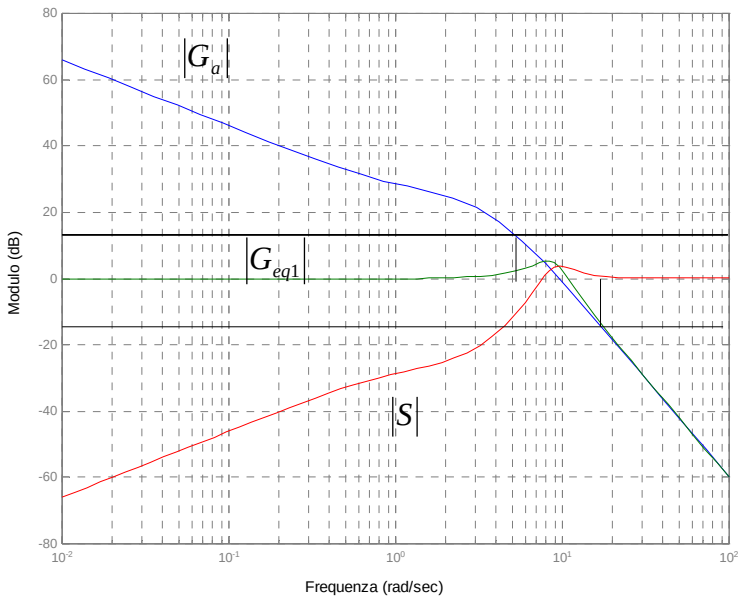


Fig. 6.3 Modula of G_{eq1} and S

The approximations (6.1) apply outside an intermediary frequency range in the neighbor of the frequency ω_c, that is called "*cross-over frequency*" as the modulus crosses the 0dB axis.

Previous results, even if very preliminarily, already show the essential elements of any feedback control, and allow to state few essential design rules:

- At low frequencies $G_{eq1} \approx 1$, hence, if the objective of the control is that the output follows with high fidelity a reference signal, this result is achieved only if the reference signal has not frequency contents over ω_c.
- Complementarily, the sensitivity is $S = 1 - G_{eq1}$, hence, in this same frequency band the sensitivity of the controlled output to disturbances is less than one, with a value that is the reciprocal of the loop gain.
- The beneficial effects of the closed loop, however, cannot be extended outside a certain region. At high frequencies the sensitivity becomes invariably one, i.e. no disturbance reduction, the feedback becomes ineffective and the closed loop behaves as the open loop.

The previous simple approximations neither offer information of the behavior of the controlled system in this intermediary frequency range, nor of the stability of the closed

loop. We will shortly see, however, that intermediary frequency range and closed loop stability are intimately linked together, as it will be clarified in the next section.

6.1.2 Closed loop stability

Obviously the control cannot ignore stability, as this is the first requirement. A natural question in frequency analysis, hence, is the following: which behavior (in frequency) of the loop transfer function is needed to guarantee stability of the closed loop?
To answer this question let rewrite the loop return transfer function

$$G_r(s) = 1 + G_a(s) = \frac{n_a(s) + d_a(s)}{d_a(s)} = \frac{d_{cc}(s)}{d_a(s)}$$

and verify that the zeros of this function are the pole of the closed loop system, and the poles are the open loop poles. Hence, the closed loop is stable if $G_r(s)$ has no zeros in the right half plane of *s*. Obviously this will be an internal stability, only if no perfect cancellations between zeroes and poles of $G_a(s)$ in the right half plane exist.

The question about stability posed at the beginning of this section is answered by the *Nyquist criterion*, which is the fundamental point of the classical approach in frequency. This criterion, with its related techniques, offers, not just a graphical method to analyze and design a stable feedback loop, but also to check sensitivity of the solution in the present of system uncertainties. From this point of view Nyquist criterion relates naturally to the "*small gain theorem*" that is the basic element of modern robust control theory.

Theorem 6.1 Nyquist criterion

A closed loop system is stable if the graph of its loop transfer function $G_a(s)$ corresponding to a continuous and close path of *s* on the complex plane, that encloses the whole right half plane moving clockwise, encircles counterclockwise the point (-1,*j*0) a number of times equal to the number of unstable poles of $G_a(s)$.

In fact, the following result can be verified

$$n_{ro} = n_{\lambda cc} - n_{\lambda a}$$

where n_{ro} is the number of clockwise rotations of the polar plot around (-1,j0), $n_{\lambda a}$ is the number of unstable poles in the open loop and $n_{\lambda cc}$ is the number of unstable poles of the closed loop.

Proof

The proof results from an important theorem of the theory of complex variables. It states that if $G(s)$ is a function assuming a unique value on and inside a contour *C* simple and closed of *s*, is analytical (no poles) and is different from zero on the contour, then

$$\frac{1}{2\pi j}\oint_C \frac{G'(s)}{G(s)}ds = n_\lambda - n_z$$

where n_λ is the number of poles and n_z is the number of zeroes of $G(s)$ inside C. It can be shown that the integral is equal to

$$\frac{1}{2\pi j}\oint_C \frac{G'(s)}{G(s)}ds = \frac{1}{2\pi}\Delta\left(\angle G(s)\right) = n_{ro}$$

the number of encirclement around the origin (multiples of 2π of change of phase of the complex function $G(s)$) of the close path generated from $G(s)$ on the polar plot when s complete C.

Let consider the loop return function $1+G_a(s)$, which zeroes are the poles of the closed loop and which poles are the poles of the open loop, noting that translating the origin to the point (-1,j0), previous results apply to the loop transfer function $G_a(s)$ and prove the theorem.

Finally, in the case of instability, the criterion indicates the number of unstable poles in the closed loop. They are

$$n_{\lambda cc} = n_\lambda - n_{ro}\,.$$

In the presence of singularities on the imaginary axes, the path of s has to be planned to avoid these singularities with detours of infinitesimal ray.

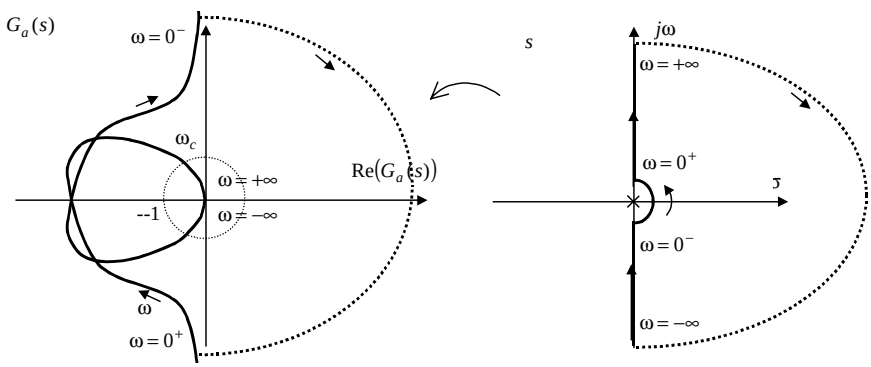


Fig. 6.4 Nyquist stability analysis

Example 6.1 The Nyquist diagram of a continuous control system

Let consider the following loop function

$$G_a(s) = \frac{3}{s(s+1)(s+2)}$$

The Nyquist diagram is shown in figure 6.5

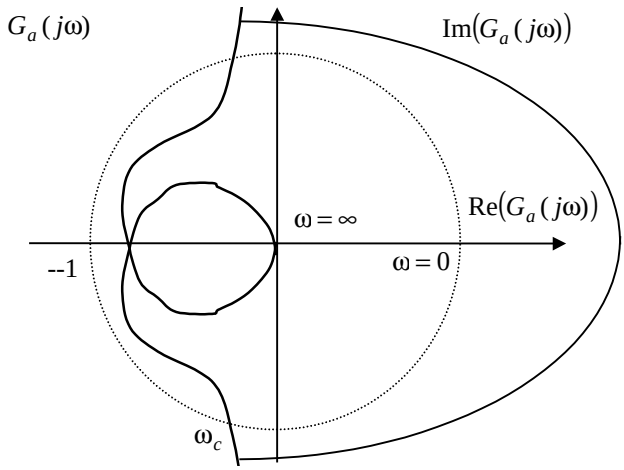


Fig. 6.5 Nyquist diagram

The closed loop in this example is stable as the critical point (-1,*j*0) is not encircled and the open loop system has not unstable poles.

Example 6.2 Nyquist diagram of a sampled data system

The same model of the previous example is adopted in a sampled data control loop with sampling period $\Delta = 1$ and ZOH filter.
The Z transform of the system is

$$G_a(z) = \frac{0.25214(z+0.1176)(z+1.91)}{(z-1)(z-0.1353)(z-0.3679)}$$

The details of the Nyquist diagrams near the point (-1, *j*0) for the continuous (continuos path) and sampled data (dashed path) systems is in figure 6.6.

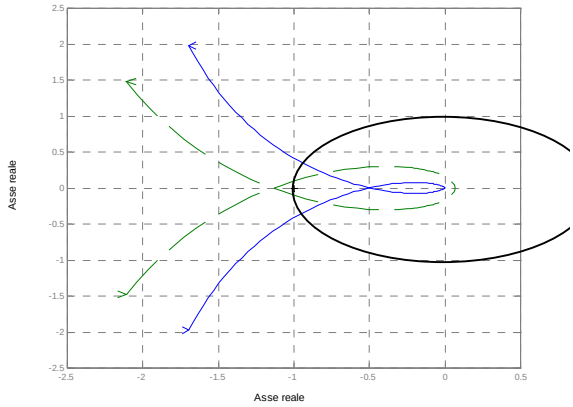


Fig. 6.6 Stability – the effect of sampling

The figure shows that sampling induces instability in a control loop originally stable in the continuous case.

The value of the Nyquist criterion was not only to test stability of the closed loop system in times when computers were not available to compute closed loop poles, but to show geometrically properties originally called *stability margins* of the control loop in the presence of uncertainty. Today these properties are referred to as robustness of the control.

"Stability margins" indicate the magnitude of the uncertainty in the plant needed to drive into instability a control system that was nominally designed stable. It is clear that if the Nyquist criterion is satisfied in nominal conditions, the loop transfer function must change of a finite amount before the system can loose stability. Moreover, to have a transition from stability to instability, a change of the number of encirclements will be needed, and necessarily for some value of ω the loop transfer function must cross the point (-1,*j*0). The interested frequencies to a possible cross-over are those that jointly verify:

$$\left|G_a(j\omega)\right| \approx 1 \text{ and } \angle G_a(j\omega) \approx (2k+1)\cdot\pi \,.$$

This is the reason why we call (-1,*j*0) "*critical point*" for the stability of the loop.

These frequencies, in the neighbor of ω_c, are the intermediary range of frequencies we introduced in the previous section and the Nyquist results will help to answer the remaining questions completing the analysis of the feedback loop over the whole frequency range.

6.1.3 The functions S and G_{eq1}

Let consider figure 6.5 where the polar plot indicates a stable closed loop system. We introduce in the coordinate plane a family of circles with center in the critical point. Their ray can be assumed as a measure of the stability margin, as it represents the distance of the open loop function from this point. Greater is this distance, greater is the magnitude of perturbations in the plant needed to drive the closed loop system in instability.

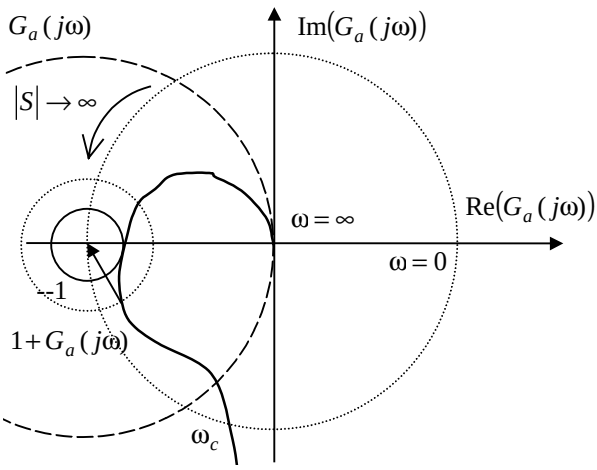


Fig. 6.7 Circles of constant modulus of the sensitivity function

For any given value of ω the vector from the loop function to the critical point is the loop return difference function, and its modulus $|1+G_a|$ is the reciprocal of the sensitivity function. Each concentric circle is the locus of points of the plane of constant modulus of the sensitivity function with value the reciprocal of the ray. The most internal circle tangent to the function indicates the distance of the loop function from the critical point, and gives, as well, the maximum of the sensitivity function.

We are in the condition to interpret the sensitivity function in the range of intermediary frequencies. Let consider in the family the unitary circle (shown dashed). In the whole range of frequencies for which the loop transfer function is inside this circle the module of the sensitivity is greater than one reaching the maximum on the tangent circle. This maximum grows to infinite when the stability margin becomes zero.

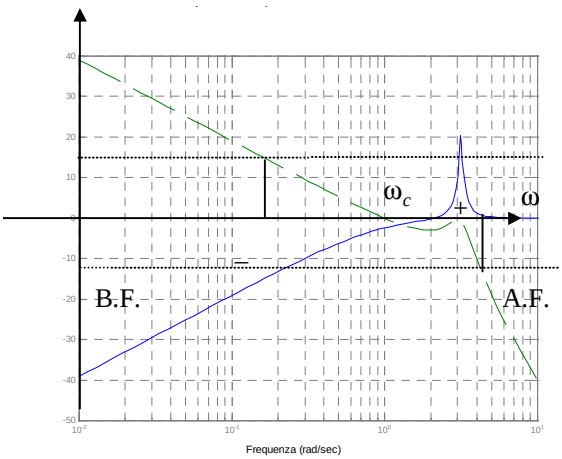


Fig. 6.8 Modulus of the sensitivity function in the intermediary frequency range

Figure 6.8 shows a very interesting result: feedback is able to reduce the effects of disturbances, but also to amplify them.
At this regard, a classical result, attributed to *Bode*, states that the difference between the areas with positive and negative sensitivity module on the Bode diagram when the open loop has at least a prevalence of two poles over the zeros, is always zero, or in the presence of unstable open loop poles is positive

$$\int_0^\infty \log\left|S(j\omega)\right| d\omega = \pi \cdot \log e \cdot \left(\sum_{\mathrm{Re}\, s>0} \mathrm{Re}\, p_i \right).$$

In conclusion the role of the feedback in the control is not just to reduce indiscriminately the sensitivity, but instead to shift the unavoidable sensitivity from sensible frequencies regions to others that the designer select as safe for the performances of the control.
Similar reasoning can be done for $G_{eq1}(j\omega)$. There exists, also for this function, a family of circles on the Nyquist diagram representing regions of constant modulus.
Let compute the modulus of $G_{eq1}(j\omega)$ expressed as function of the real and imaginary parts of $G_a(j\omega)$

$$G_{eq1}(j\omega) = \frac{G_a(j\omega)}{1+G_a(j\omega)}; \quad G_a(j\omega) = R + j \cdot I$$

$$\left|G_{eq1}(j\omega)\right|^2 = \frac{R^2 + I^2}{(1+R)^2 + I^2} = M^2. \tag{6.1}$$

Equation (6.1) represents a family of circles, function of the parameter M, the modulus of $G_{eq1}(j\omega)$:

$$\left(R-\frac{M^2}{1-M^2}\right)^2+I^2=\left(\frac{M}{1-M^2}\right)^2$$

with center and ray given by:

$$c=\left(\frac{M^2}{1-M^2},0\right),\quad r=\frac{M}{\left|M^2-1\right|}.$$

These circles degenerate when M=1 in the vertical line x=-1/2, when M goes to infinite they accumulate, with ray becoming zero, around (-1,j0), and when M goes to 0 around the origin.

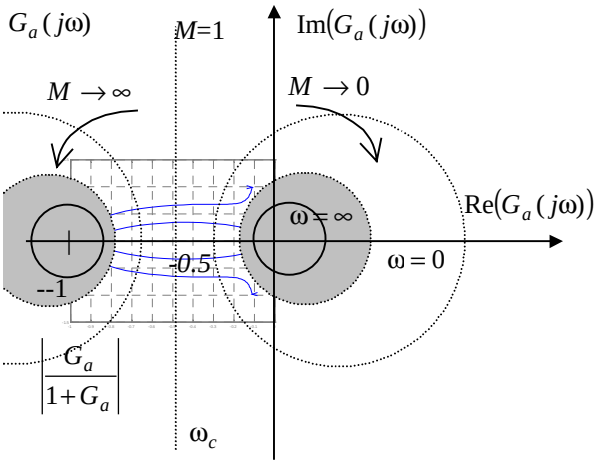


Fig. 6.9 Circles of constant $\left|G_{eq1}(j\omega)\right|$

As for the sensitivity, when the loop function is near the critical point the modulus of $G_{eq1}(j\omega)$ becomes large, reaching the maximum for the frequency of tangency of the loop function with the most internal circle. Hence, this function has a maximum greater than one any time the polar diagram crosses the vertical line with abscissa -0.5. In this case the closed loop transfer function shows a *resonance peak*, that grows to infinite when the stability margins decrease to zero.

Example 6.3 Functions S and G_{eq1} in an example

Figure 6.10 draws the module of the functions G_a, S, G_{eq1} of example 6.1.

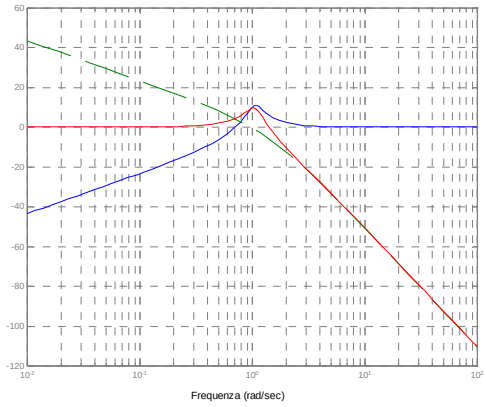


Fig. 6.10 Modula of the transfer functions in open and closed loop

The next section shows the classical techniques adopted in the literature to analyze stability margins, that we present for historical reasons.

Stability margins and Nichols chart

Once stability has been guaranteed with an appropriate choice of $G_a(j\omega)$, the closed loop behavior will depend on the distance of the open loop function from the critical point and, hence, from the stability margins.
Traditionally to cope with complex functions two indices were adopted to indicate stability margins: phase margin and gain margin. They are justified if we assume that the system has independent perturbations on the phase and on the gain of its transfer function, e.g. in the first case the presence of an unknown delay or, in the second case, the change of the gain of an amplifier.

Definition 6.1 Phase margin

Phase margin is the perturbation of phase ψ needed by the loop transfer function to drive the closed loop system from the stable nominal condition to instability. This phase is also the difference between π and the loop phase at the frequency where modulus is 1.

Definition 6.2 Gain Margin

Gain margin m is the reciprocal of the amplification factor needed by the loop function to drive from stable nominal conditions the closed loop to instability. This value is the modulus of the loop function when its phase is π.

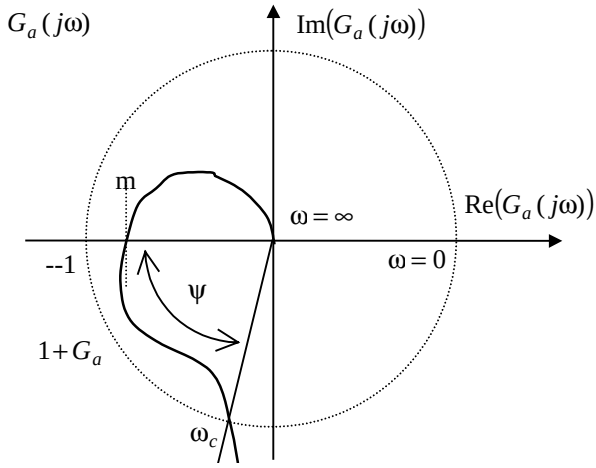


Fig. 6.11 Phase and gain margins

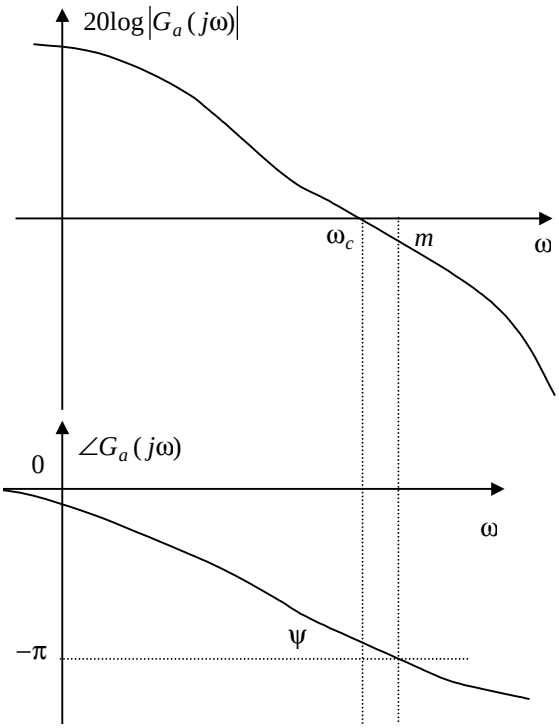


Fig. 6.12 Phase and gain margins on the Bode diagrams

The path of the loop function on the Nyquist plot can differ form the basic one used in the previous examples (see figure 6.17 and 6.18), especially when poles or zeroes in the right half plane are present. In these conditions, once the encirclements of the Nyquist criterion have been verified, stability margins have to be interpreted as the absolute minimum perturbation, either in positive/negative phase shift or amplification/attenuation of the gain, that brings the loop function over the critical point. With the caution that encirclements can be judged only on the Nyquist diagram, the numerical values of the margins can be read more easily on the Bode diagrams.
An alternative technique, equivalent to the Bode diagram, that merges in a unique plot both phase and gain margins with a certain popularity is called *Nichols chart*.

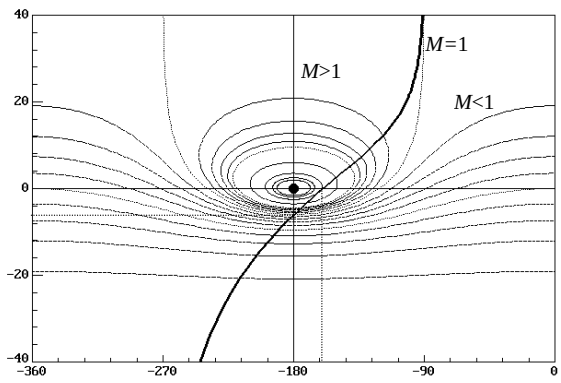


Fig. 6.13 Nichols chart

The Nichols chart presents on the abscissa the phase and on ordinate the modulus in logarithmic scale of the loop transfer function in frequency. The critical point is in the center of the graph at the origin of the coordinates. Loop gain perturbation translates in a vertical shift of the graph, while phase perturbation in an horizontal shift. Phase and gain margins are read at the intersection of the graph with abscissa and ordinate axes. Circles of constant modulus of G_{eq1} of the Nyquist diagram become here the curves shown in figure 6.13. The curve with M=1, separating values lower and greater than one is clearly visible. Plotting the M values at the intersections as function of ω offers $\left|G_{eq1}\right|$. To obtain the sensitivity function using the same family of curves observe that:

$$F = \frac{1}{G_a}$$

$$S = \frac{F}{1+F} .$$

In other words, the sensitivity function is obtained from the M circles by plotting on the Nichols chart the inverse of the loop transfer function. Because of a symmetry with respect to the vertical axes it is sufficient to plot symmetrically with respect to the horizontal axis the loop function.

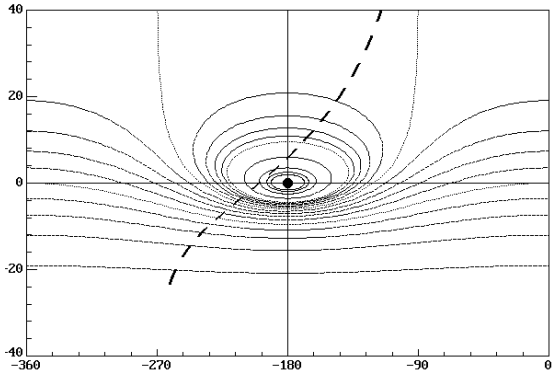


Fig. 6.14 Computing S from $F = \dfrac{1}{G_a}$

6.2 Plant uncertainty and robust stability

Several attempts have been done to generalize previous results, especially referred to specific parameter variations. The most significant author of this tendency was certainly *Horowitz*. However, results and techniques never received a general acceptance, in spite of the author achievements in the professional activity. The reason for this has been already explained at the beginning: techniques are difficult to apply, do not exist formal results that allow generalizations, these cannot be translated in numerical algorithms.
Robust control theory, with the description of uncertainty given in chapter **5** overcomes previous problems, and shortly we show is a generalization of classical results.
Let consider the description of uncertainty in multiplicative form, referring to a SISO loop transfer function:

$$G_p = (I + \Delta \cdot W_2) \cdot G$$

$$\left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right| = \left| \frac{\Delta_G(j\omega)}{G(j\omega)} \right| = |\Delta(j\omega) \cdot W_2(j\omega)| \le |W_2(j\omega)| ,$$

This is equivalent to envision a region of uncertainty enveloping the nominal polar plot represented by circles of ray $|W_2(j\omega) \cdot G(j\omega)|$.

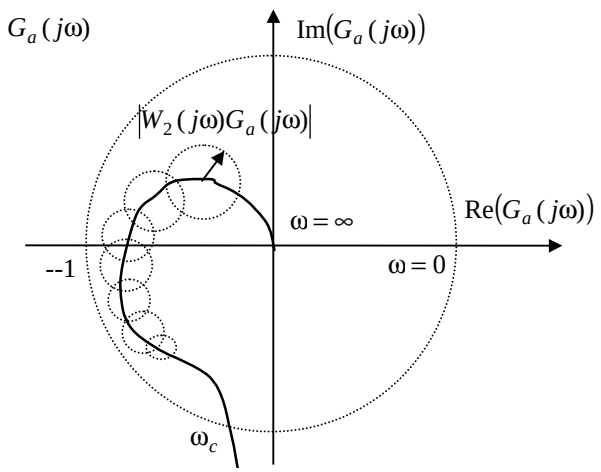


Fig. 6.15 Uncertainty region of the loop transfer function

The condition for robust stability

$$\sup_\omega \left|W_2(j\omega)\cdot G_{eq1}(j\omega)\right| = \left\|W_2\cdot G_{eq1}\right\|_\infty < 1$$

has a clear geometrical interpretation as it guarantees that the region of uncertainty for any ω doesn't encompass the critical point, i.e.:

$$\left|W_2(j\omega)\cdot G_a(j\omega)\right| \le \left|1+G_a(j\omega)\right| .$$

This result clearly relates the Nyquist criterion to the "*small gain theorem*" seen in the previous chapter.
The next figure shows a typical uncertainty mask for a dynamical system.

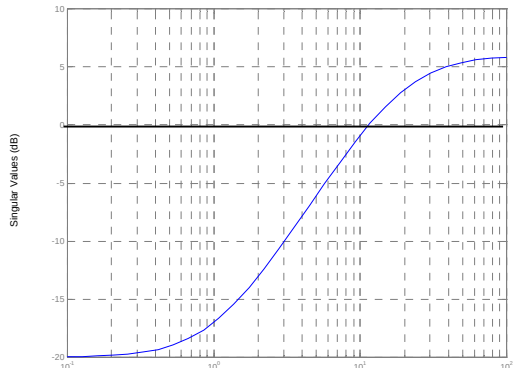


Fig. 6.16 Typical shape of the uncertainty in a dynamical system

A 10% variation of the steady state gain is evident at low frequencies; then dynamic uncertainty grows with the frequency, until when crossing the 0 dB axis a relative uncertainty greater than one indicates when the selected model becomes completely unreliable.

6.2.1 Gain and phase margins as unstructured uncertainty

It is interesting to interpret the classical concepts of phase and gain margin previously defined using robust stability results. Let assume that the plant uncertainty is represented by a changing gain. We define the weighting function as follows:

$$\frac{\tilde{G}_a - G_a}{G_a} = \Delta \cdot W_2$$

we obtain

$$\tilde{G}_a = \tilde{k} \cdot G, \quad k_m \le \tilde{k} \le k_M ,$$
$$k_n = \frac{k_M + k_m}{2}, \quad W_2 = \frac{k_M - k_n}{k_n}$$
$$G_a = k_n \cdot G, \quad \tilde{G}_a = (1 + \Delta W_2) \cdot G_a$$

Alternatively we image that only the phase of the plant changes:

$$\tilde{G}_a = e^{j\tilde{\theta}} \cdot G, \quad -\theta_M \le \tilde{\theta} \le +\theta_M .$$

The uncertainty model is the following:

$$\Delta \cdot W_2 = e^{j\tilde{\theta}} - 1, \quad W_2 = 2 \cdot sin \frac{\theta_M}{2} .$$

In both cases the weighting function W_2 is a constant and the condition of robust stability requires

$$\left|W_2 \cdot G_{eq1}\right| \le 1, \quad \left|G_{eq1}\right| \le \frac{1}{W_2}.$$

This defines the limiting circle for stability with parameter $M = \dfrac{1}{W_2}$.

It is interesting to compare this result with the classical criterion based on phase and gain margins.

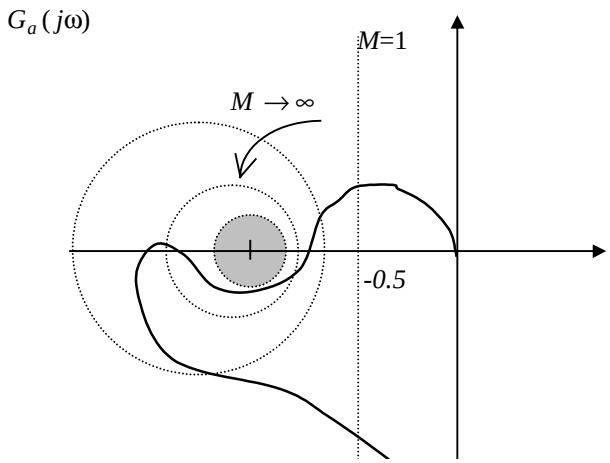


Fig. 6.17 Gain variations translated to multiplicative uncertainty

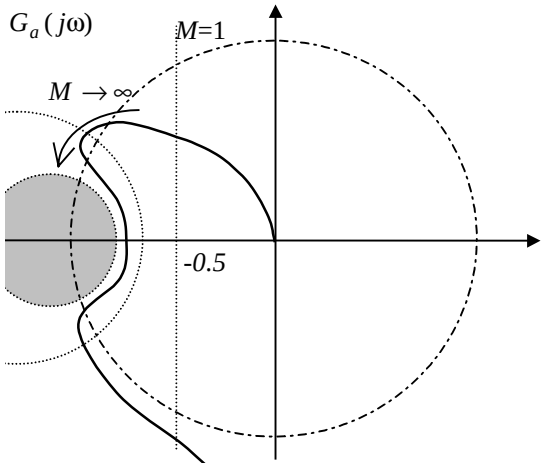


Fig. 6.18 Phase variations translated to multiplicative uncertainty

6.3 Time response of a closed loop system

The frequency analysis of the loop transfer function has been aimed so far to establish stability, stability margins, and sensitivity of the closed loop to disturbances. The dynamic behavior of the closed loop system response to a reference signal is, however, also part of the requirements. This section outline the role that the classical closed loop functions $G_a(j\omega), S(j\omega)$ and $G_{eq1}(j\omega)$ have in characterizing the response to deterministic signals. The problem is approached at steady state and during the transient.

6.3.1 Steady state behavior

Steady state behavior in reference-output or disturbance-output response is one of the first requirements, after stability margins, that interests a feedback control system. This analysis is conducted analyzing tracking error or disturbance error to polynomial reference signals. We approach here tracking error, as disturbance error is very similar. Let be $G_{eq}(s)$ the reference-output transfer function of the closed loop system. No hypothesis have been made yet on the characteristic of the loop, so $G_{eq}(s)$ may not coincide with $G_{eq1}(s)$.

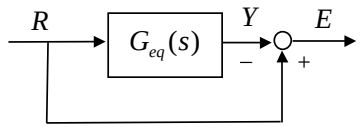


Fig. 6.19 Reference-output error

Output and error are:

$$y(s) = G_{eq}(s) \cdot r(s)$$

$$G_{eq}(s) = \frac{n_{eq}(s)}{d_{eq}(s)} = \frac{k_{cc}^{*} \cdot \prod_{i=1}^{m}(s - z_{cci})}{\prod_{i=1}^{n}(s - \lambda_{cci})} = \frac{b_{ccm} \cdot s^{m} + \cdots + b_{cc1} \cdot s + b_{cc0}}{s^{n} + \cdots + a_{cc1} \cdot s + a_{cc0}}$$

$$b_{ccm} = k_{cc}^{*}; \quad b_{cc0} = k_{cc}^{*} \cdot \prod - z_{cci}; \quad b_{cc1} = b_{cc0} \cdot \frac{1}{-z_{cci}};$$

$$b_{cc2} = b_{cc0} \cdot [(\frac{1}{-z_{cci}})^{2} - \frac{1}{z_{cci}^{2}})]$$

$$e(s) = r(s) - y(s) = (1 - G_{eq}(s)) \cdot r(s)$$

The steady state error, in particular, is derived when the reference is of polynomial type, e.g. unitary step, ramp and parabola:

$$r(s) = \frac{1}{s^{i}}, \; i = 1,2,3$$

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} s \cdot e(s) = \lim_{s \to 0} \frac{s^{n} + \cdots + (a_{cc1} - b_{cc1}) \cdot s + (a_{cc0} - b_{cc0})}{s^{n} + \cdots + a_{cc1} \cdot s + a_{cc0}} \cdot \frac{1}{s^{i-1}}$$

The next table shows the asymptotic tracking errors in the different conditions

ref. \ a_{cci}, b_{cci}	$a_{cc0} \neq b_{cc0}$	$a_{cc0} = b_{cc0}$	$a_{cc1} = b_{cc1}$
step	$\frac{a_{cc0} - b_{cc0}}{a_{cc0}}$	0	
ramp	∞	$\frac{a_{cc1} - b_{cc1}}{a_{cc0}}$	0
parabola	∞	∞	$\frac{a_{cc2} - b_{cc2}}{a_{cc0}}$

The results can be linked to the zeros and poles of *Geq*(*s*).
Remembering the relationship between coefficients and roots of a polynomial

$$a_{cc0} = \prod -\lambda_{cci}; \quad b_{cc0} = b_{ccm} \cdot \prod -z_{cci}$$

$$a_{cc1} = \prod -\lambda_{cci} \cdot \frac{1}{-\lambda_{cci}}; \quad b_{cc1} = b_{ccm} \cdot \prod -z_{cci} \cdot \frac{1}{-z_{cci}}$$

$$a_{cc2} = \prod -\lambda_{cci} \cdot \underset{i,j;i\neq j}{\frac{1}{\lambda_{cci} \cdot \lambda_{ccj}}}; \quad b_{cc2} = b_{ccm} \cdot \prod -z_{cci} \cdot \underset{i,j;i\neq j}{\frac{1}{z_{cci} \cdot z_{ccj}}}$$

We establish here results and needed conditions:

- The first condition imposes zero steady state error to a step reference, and a finite error to the ramp

$$a_{cc0} = b_{cc0} \rightarrow \prod -\lambda_{cci} = b_{ccm} \cdot \prod -z_{cci} \text{ unitary steady state gain} \tag{6.2}$$

- The second result, when the first condition is satisfied, is the steady state ramp error

$$\frac{a_{cc1} - b_{cc1}}{a_{cc0}} = \frac{1}{-\lambda_{cci}} - \frac{1}{-z_{cci}} \tag{6.3}$$

- The third result gives, when the ramp error is zero, the steady state error to a parabola

$$\frac{a_{cc2} - b_{cc2}}{a_{cc0}} = \underset{i,j;i\neq j}{\frac{1}{\lambda_{cci} \cdot \lambda_{ccj}}} - \underset{i,j;i\neq j}{\frac{1}{z_{cci} \cdot z_{ccj}}} = \frac{1}{z_{cci}^{\ 2}} - \frac{1}{\lambda_{cci}^{\ 2}} \tag{6.4}$$

We see that zeros and poles imposed, in the design, to the closed loop transfer function *Geq*(*s*), not only characterize the dynamics, but also the steady state errors. These properties don't dependent directly from the loop characteristics, so can be object of a separate design.
As an exercise we discuss now the choice of the closed loop zeros and poles to satisfy desired properties in simple cases of interest.
The simplest case is to describe the unitary closed loop transfer function as a *pair of dominant poles*, of type:

$$G_{eq}(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega^2{}_n}$$

This is a common approximation of a closed loop response where unitary gain has been guaranteed.
This function has a steady state error to the ramp of:

$$e_\infty = \frac{2\zeta}{\omega_n}. \tag{6.5}$$

It is immediate to verify that this type of function will never be able to guarantee zero error to the ramp or finite error to the parabola. For this, at least one zero must be added in the transfer function. A classical solution is to add a pair zero-pole such as:

$$G_{eq}(s) = \frac{\omega_n{}^2 m(s+z)}{(s^2 + 2\zeta\omega_n s + \omega^2{}_n)(s+mz)} \tag{6.6}$$

The steady state gain is unitary, and the tracking error

$$e_\infty = \frac{2\zeta}{\omega_n} + \frac{1}{mz} - \frac{1}{z} = \frac{2\zeta}{\omega_n} - \frac{m-1}{mz} \tag{6.7}$$

can be reduced to zero with a proper choice of *m* and *z*.
Moreover, if zero and pole are positioned at low frequency, relatively near one to the other they will not influence significantly the dominant response due to the pair of complex conjugate poles.

Unitary feedback loop

As a particular case $G_{eq} \equiv G_{eq1}$ is the result of a unitary feedback loop, so properties of the closed loop can be transferred to the open loop function. The system is represented in the next figure:

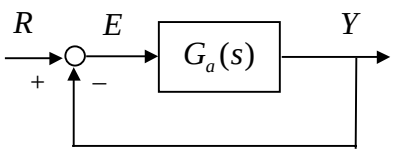


Fig. 6.20 Reference-output error in unitary feedback loop

$$1 - G_{eq1} = S = \frac{1}{1+G_a} = \frac{s^n + \cdots + a_{ar} \cdot s^r}{s^n + \cdots a_{cc1} \cdot s + a_{cc0}}$$

$$\lim_{s\to 0} \frac{1}{1+G_a} = \lim_{s\to 0} \frac{s^r}{s^r + k_r}, \quad k_0 \equiv k_{st}, \quad k_1 \equiv k_v, \quad k_2 \equiv k_a$$

where *r* is the number of pole at the origin present in the loop function and k_i is the loop steady state gain. A control loop is defined of *type r* according to the number of poles at the origin present in the loop function, hence type zero loop has not poles at the origin, and so on. The following table gives the steady state error as function of the degree of the polynomial of the reference signal and the type of the control loop.

ref. \ type	0	1	2
step	$\frac{1}{1+k_{st}}$	0	
ramp	∞	$\frac{1}{k_v}$	0
parabola	∞	∞	$\frac{1}{k_a}$

The results show that the steady state behavior is defined by the number of poles at the origin present in the open loop.

Non unitary feedback loop

If the feedback loop is not unitary, but has a dynamics, it is possible to guarantee asymptotic performances to the tracking error without poles at the origin in the loop function.

The system can be expressed as unitary feedback loop and a forward function $\frac{1}{H(s)}$:

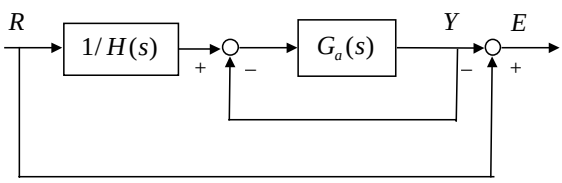


Fig. 6.21 Reference-output error in non-unitary control loops

$$H(s)=\frac{\cdots+b_{h2}\cdot s^2+b_{h1}\cdot s+b_{h0}}{\cdots+a_{h0}}$$

$$e(s)=(1-\frac{G_a(s)}{(1+G_a(s))\cdot H(s)})\cdot r(s)$$

$$e(s)=\frac{(1+G_a(s))\cdot H(s)-G_a(s)}{(1+G_a(s))\cdot H(s)}\cdot r(s)$$

$$e(s)=\left(\frac{1}{1+G_a(s)}-\frac{G_a(s)}{1+G_a(s)}\cdot\frac{1-H(s)}{H(s)}\right)\cdot r(s)$$

Without pole at the origin in the loop error zero to the step is guaranteed by a proper gain of the feedback:

$$\lim_{s\to 0}\frac{H(s)}{1-H(s)}=\lim_{s\to 0}G_a(s)$$

$$\frac{b_{h0}}{a_{h0}-b_{h0}}=\frac{b_{a0}}{a_{a0}}$$

$$k_{sth}=\frac{k_{sta}}{1+k_{sta}}$$

In this case the error to the ramp:

$$e(s)=\left(\frac{\cdots+a_{a1}\cdot s}{\cdots+a_{a0}+b_{a0}}-\frac{\cdots+b_{a1}\cdot s}{\cdots+a_{a0}+b_{a0}}\cdot\frac{\cdots+(a_{h1}-b_{h1})\cdot s}{\cdots+b_{h0}}\right)\cdot\frac{1}{s^2}$$

$$e_\infty=\frac{a_{a1}}{a_{a0}+b_{a0}}-\frac{b_{a1}}{a_{a0}+b_{a0}}\cdot\frac{a_{h1}-b_{h1}}{b_{h0}}$$

can become zero if:

$$\frac{(1+k_{sta})\cdot\dfrac{1}{-\lambda_{ai}}}{\dfrac{1}{-z_{ai}}}=\frac{1}{-\lambda_{hi}}-\frac{1}{-z_{hi}}. \tag{6.8}$$

In conclusion it is possible to control the asymptotic error if precise condition are satisfied from poles and zeros of the feedback link.
There is, however, a fundamental difference between results given by the presence of poles at the origin in unitary feedback, and these ones:

- In the first case results are guaranteed also in the presence of plant variations;
- In the second case results are valid only in nominal conditions.

Example 6.4 *Feedforward"* control

Previous results explain a classical technique used in the numerical control of machine tools were error zero to the ramp is obtained in a loop with only one pole at the origin, by simply adding a forward block on the reference.
Let be the closed loop function of a given control loop given by

$$G_{eq}(s)=\frac{\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega^2{}_n}$$

with a steady state tracking error given by:

$$e_\infty = \frac{2\zeta}{\omega_n}.$$

We estimate the speed of the reference using the following simple filter

$$G_f(s) = \frac{\lambda \cdot s}{s+\lambda}$$

and we introduce a dynamic block in cascade to $G_{eq}(s)$ to process the reference signal

$$H^{-1}(s) = 1 + h \cdot G_f(s) = \frac{s+\lambda+h\cdot\lambda\cdot s}{s+\lambda} = \frac{(1+h\cdot\lambda)\cdot s+\lambda}{s+\lambda}$$

This is equivalent of to a non-unitary feedback. If we select *h* satisfying (6.8), i.e.

$$h = \frac{2\zeta}{\omega_n},$$

the asymptotic tracking error will be zero.

6.3.2 Transient response

The simplest requirements during the transients are those assigned to the time response of the output to a unitary step. Four aspects of the response are subject to specifications:

- Delay time;
- Rising time;
- Overshoot,
- Settling time.

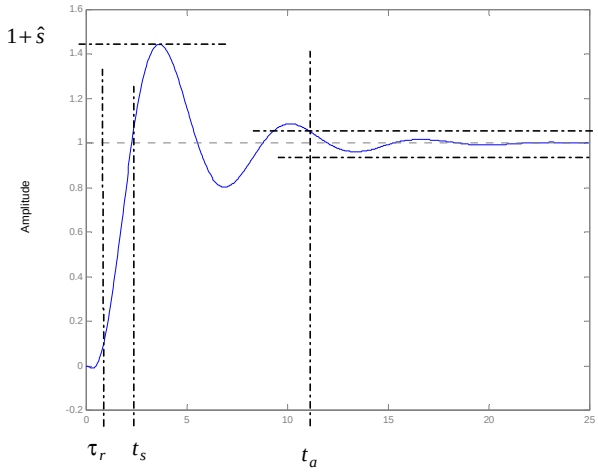


Fig. 6.22 Time requirements in the transient

Delay time τ_r

The time needed by the output to reach a significant value greater than zero (e.g. 5% of the steady state).

Rising time $t_s - \tau_r$

The time, after the delay, to reach for the first time the steady state value.

Overshot $\hat{s}$

The difference between the maximum value reached during the transient and the steady state.

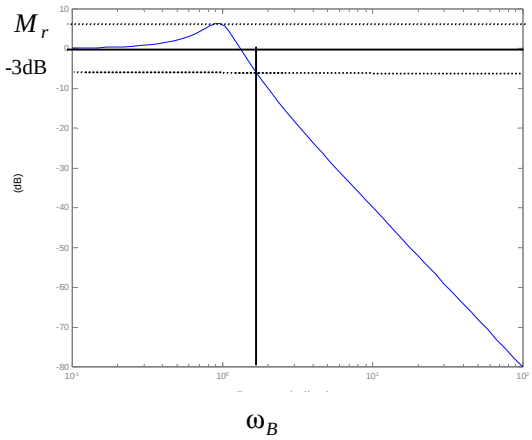
Settling time t_a

The total time required to the output to enter definitively into a given tolerance band around the steady state (e.g. $\pm 0.5\%$).

With a certain degree of approximation two essential frequency and time requirements can be linked each other. These results derive from a usually satisfactory approximation of the dominant poles of the closed loop response to a second order system.
The delay time is a consequence of the delay naturally present in the plant (or by right half plane zeros)[11] and the designer cannot do much to change it; settling time will be

[11] A link between delay and right half plane zeros is detected if you consider that a simple delay approximation has one zero and one pole at the same frequency and in opposite half planes.

conditioned by the closed loop poles at the lowest frequencies and will be discussed



later during the design.

Fig. 6.23 Module of the closed loop transfer function

The essential characteristics of the closed loop transfer function, assuming steady state gain equal to one, are: the frequency band, the frequency at which the gain is –3 dB below the steady state, and the resonance peak.
Frequency band and raising time are linked by

$$\omega_B \cdot t_s \approx 3 .$$

Resonance peak of the transfer function and overshot are linked by

$$1 + \hat{s} \approx 0.85 \cdot M_r ,$$

with M_r expressed in absolute value and not in dB.
In a unitary feedback system, requirements on G_{eq1} translate directly to requirements on the sensitivity function. In fact, let assign a given $G_{eqdes}(s)$, we get:

$$S_{des}(s) = 1 - G_{eqdes}(s)$$

and assume a weighting function:

$$|W_1(j\omega)| = |S_{des}(j\omega)|^{-1} . \tag{6.9}$$

In the loop design, if we satisfy the following performance index

$$|W_1(j\omega) \cdot S(j\omega)| \le 1, \forall \omega$$

we have a bound on the sensitivity function

$$|S(j\omega)| \le |W_1(j\omega)|^{-1}, \forall \omega \cdot$$

This will guarantee that the resulting G_{eq1} will outperform specification in terms of frequency band and resonance peak.
A typical shape of the weighting function W_1 for guaranteeing unitary feedback closed loop requirements will be

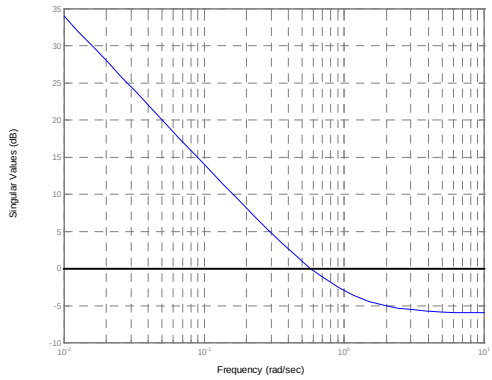


Fig. 6.24 Typical shape of the sensitivity weighting function W_1

This shape embeds three significant aspects: it ensures zero steady state tracking error to the step, a minimum guaranteed stability margin to the loop, a minimum guaranteed frequency band and a maximum resonance peak to $G_{eq1}(s)$.

Example 6.5 Design of a $G_{eq1}(s)$ with a given rising time and overshoot

Let impose the following closed loop requirements: rising time $t_s \le 0.1$ overshoot $1+\hat{s} \le 1.35$. From previous approximations we derive $\omega_B \ge 30$ and $M_r \le 1.58$.
Adopting as representative a second order system we assume the desired closed loop function

$$G_{eqdes}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad S_{des}(s) = \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta = 0.35 \quad \omega_n = 20$$

A possible function W_1 which reciprocal is a bound for $S_{des}(s)$ is

$$W_1 = \frac{s+10}{2s} \cdot$$

Let compare the responses of two different closed loop systems satisfying the sensitivity requirements. The first is exactly G_{eqdes}, the second has closed loop transfer function

$$G_{eq}(s) = \frac{500\,(\mathrm{s}+5)}{(\mathrm{s}+6.605)\,(s^2+18.39\mathrm{s}+378.5)}$$

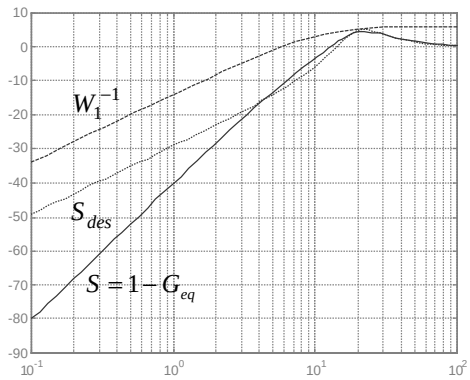


Fig. 6.25 Reciprocal of the weighting function W_1 and sensitivities

The step time responses of the two functions are presented in the next figure

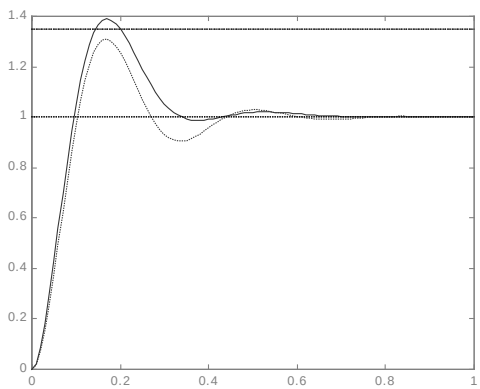


Fig. 6.26 Time response and limits of the overshoot

6.4 The design of a closed loop system

6.4.1 Requirements

If all previous considerations are collected we can express a set of requirements for a feedback control system. These requirements will form the basis for the design. In the following the original classical style has already been influenced by modern robust control techniques with the introduction of weighting functions. A similar approach can be found in reference [6].
In logical and importance order these are the requirements:

1. Stability margins to multiplicative disturbances – alternatively interpreted as robust stability to plant uncertainties;
2. Tracking errors to reference signals at steady state;
3. Output sensitivity to additive disturbance;
4. Transient response:
    - damping of the (dominant) closed loop poles;
    - rising time (closed loop frequency band);
    - transient overshoot;
    - settling time;
5. Sensitivity of the closed loop reference-output transfer function to plant uncertainties;
6. Control activity as consequence of measurement noises;
7. Control activity as consequence of a transient of the reference.

Point 1, we already know, involves $G_{eq1}(s)$, the unitary feedback closed loop function. Points 2, 3 and 5 pose constraints to the sensitivity function from steady state up to the loop frequency band w_c. Point 4, in one degree of freedom design, translates to a condition on the sensitivity function, otherwise it involves the second degree of freedom. Points 6,7 add constraints to the compensation filter.
Obviously requirements are not always consistent:
Constraints on the sensitivity are represented by the weighting function W_1. We will see that in the presence of zeros and poles of the plant in the right half plane not any weighting function can be assumed.
Stability margins to multiplicative noises are expressed by the weighting function W_2, and this may conflict against W_1.
Control activity is expressed through the weighting function W_3, which has on the project effects similar to W_2.
Requirements on performance and robustness usually conflict each other, primarily why high performances ask for high loop gains, and in the presence of uncertainties with high gain is difficult to guarantee robustness.
High control activity is the cost the designer has to pay for high performance controls.
Finally, requirements on time response (frequency band) may not take into account sensitivity of the closed loop transfer function to plant uncertainty.

The design, as usual in engineering, will be the result of a compromise where the weighting functions W_1 and $W_2 - W_3$ will play antagonistic roles.

Two different steps should be considered in designing a control:

1. The selection of the input signals to control the plant and of the output measurements used for feedback (if more than one is used);
2. Then, the design of the control filters.

The problems of adopting more than one input and output involves multivariable controls. We know that the classical approach is not suitable for this, and with one exception discussed later, it will be discussed in the next chapter.
For selecting the control architecture, the starting point for a design is to check requirements 1-3 and 7. If these requirements are consistent, specifically in the relationship between S and G_{eq1}, the design can be performed using a simple control in cascade with "one degree of freedom architecture". Otherwise, the control loop requires two blocks and we approach a "two degrees of freedom design".
A one degree of freedom design, with objective the loop characteristics is, however, the starting point in both cases.
In fact when needed, with the loop function solved to approach sensitivity, the design is extended to the two degrees of freedom which involves the desired reference-output transfer function.
In the next section we outline the classical approach of tentative design where filters are first selected to satisfy some of the specifications, then from the analysis of the whole envelop of requirements are modified with successive iterations.

6.4.2 Performance and robustness

Previous requirements are translated in three weighting functions W_1, W_2, and W_3 used to generate a condition of robust performance function, seen in chapter 5, explicitly of the sensitivity and the unitary closed loop transfer function:

$$\left|W_1\left(I+GG_c\right)^{-1}\right| + \left|W_2 \cdot GG_c\left(I+GG_c\right)^{-1}\right| \le 1 \quad \forall\omega \tag{6.10}$$

- Reduce sensitivity of additive disturbances on the output;
- Reduce the tracking error;
- Reduce sensitivity of the closed loop function to the plant uncertainties;

$$\max\left\{\left|W_1\left(I+GG_c\right)^{-1}\right|, \left|W_3 G_c\left(I+GG_c\right)^{-1}\right|\right\} + \left|W_2 \cdot GG_c\left(I+GG_c\right)^{-1}\right| \le 1, \quad \forall\omega, \tag{6.11}$$

- Add to previous requirements a constraint on the control activity.

Assignment of the sensitivity

One of first requirements of the feedback is the achievement of a desired sensitivity function

$$\left|S(j\omega)\right| < \left|W_1(j\omega)\right|^{-1}, \forall\omega$$

this is equivalent to impose a bound to the H_∞ norm:

$$\left|W_1(j\omega)\cdot S(j\omega)\right| < 1, \forall\omega \equiv \left\|W_1\cdot S\right\|_\infty < 1$$

The next is a typical requirement with absolute value of the sensitivity below *0.01* before the frequency ω_1, and after bounded by 2:

$$W_1(s) = \frac{0.5\left(s+10\cdot\sqrt{2}\cdot\omega_1\right)^2}{(s+\omega_1)^2}.$$

Constraints on the sensitivity of the closed loop transfer function

During reference tracking it is required to maintain the closed loop behavior insensitive to plant variations with a certain degree of accuracy. Let be $\overline{W_1}$ the desired weighting function and W_2 the mask of the plant multiplicative uncertainty. From known results we can write

$$\left\|\overline{W_1}\cdot\frac{\Delta_{G_{eq}}}{G_{eq}}\right\|_\infty = \left\|\overline{W_1}\cdot\frac{1}{1+\tilde{G}_a}\cdot\frac{\Delta_{G_a}}{G_a}\right\|_\infty = \left\|\frac{\overline{W_1}\cdot\Delta\cdot W_2\cdot S}{1+\Delta\cdot W_2\cdot G_{eq1}}\right\|_\infty$$

$$\leq \left\|\frac{W_1\cdot S}{1-\left|W_2\cdot G_{eq1}\right|}\right\|_\infty < 1, \quad W_1 = \overline{W_1}\cdot W_2$$

This requirement translates into a condition of robust performance

$$\left|W_1\cdot S\right| + \left|W_2\cdot G_{eq1}\right| \leq 1, \quad \forall w$$

Constraints on the control activity

Remembering note **4.2** of chapter **4**, the consequence of satisfying other control requirements may result in a G_{equ} with unacceptable high gains, especially at high frequencies. To account for this, an additional constraint must be imposed in the performance:

$$\max\left\{\left|W_1S\right|, \left|W_3G_{equ}\right|\right\} + \left|W_2\cdot G_{eq1}\right| \leq 1, \quad \forall\omega, \tag{6.12}$$

that can be rewritten as

$$\max\left\{\left|W_1S\right|, \left|W_3G^{-1}GG_{equ}\right|\right\} + \left|W_2\cdot G_{eq1}\right| \leq 1, \quad \forall\omega, \tag{6.13}$$

resulting in

$$\max\left\{\left|W_1 S\right|, \left|\overline{W}_3 G_{eq1}\right|\right\} + \left|W_2 \cdot G_{eq1}\right| \le 1, \quad \forall\omega \quad \overline{W}_3 = W_3 \cdot G^{-1}. \tag{6.14}$$

6.4.3 Achievable performances

W_{1-3} and W_2 are the three weighting functions representing desired performances and plant uncertainty. The requirements play customarily antagonist roles in the design: improving disturbance insensitivity asks for increasing the gain and widening the loop frequency band, but this contrasts with plant uncertainty which usually grows at high frequencies. Before starting the design it is important to evaluate the achievable limits, and eventually modify the control requirements.

We already know that sensitivity cannot be reduced on a arbitrary large frequency band, moreover sensitivity and unitary feedback closed loop function are linked by the following relationship:

$$\begin{aligned} &S(s) + G_{eq1}(s) = 1, \forall s \\ &1 = \left|S(s) + G_{eq1}(s)\right| \le \left|S(s)\right| + \left|G_{eq1}(s)\right| \end{aligned} \tag{6.15}$$

As a consequence of (6.15) it is immediate (see reference [7]) to show that robust performance is achievable only if

$$\min\{\left|W_1(j\omega)\right|, \left|W_2(j\omega)\right|\} < 1, \forall\omega. \tag{6.16}$$

This means that at no frequencies both modules of the two functions can be greater than one.

Typically the module of W_1 is greater than one at low frequencies to guarantee good control characteristics, and the module of W_2 is greater than one at high frequencies to take into account un-modelled high frequencies dynamics.

Moreover if a constraint on the control activity is present, the sum of the two modules must never be greater than one

$$\left|\overline{W}_3(j\omega)\right| + \left|W_2(j\omega)\right| < 1, \forall\omega : G_{eq1}(j\omega) \cong 1$$

in the whole frequency band desired for the closed loop.

A second aspect to consider is the presence of zeros or poles of the plant in the right half plane. Control practitioners know that unstable plants, or plants with zeros on the right half plane are more difficult to control than others, inequality (6.16) is responsible for that.

Let assume that the plant has either a pole p o a zero z in the right half plane, remember that the open loop poles are zeros of the sensitivity and plant zeros are also zeros of the unitary closed loop function. It is immediate to verify that:

$$\begin{aligned} S(p) = 0 \quad &\text{hence} \quad G_{eq1}(p) = 1 \\ S(z) = 1 \quad &\text{hence} \quad G_{eq1}(z) = 0 \end{aligned}$$

From the maximum module theorem, stating that the module of a stable transfer function in the right half plane reaches it maximum on the imaginary axis, the following additional constraints for the weighting functions W_1 and W_2 result:

$$1 > \left\| W_1 \cdot S \right\|_\infty \geq \left| W_1(z) \right|$$
$$1 > \left\| W_2 \cdot G_{eq1} \right\|_\infty \geq \left| W_2(p) \right|$$

Hence the module of W_1 cannot be greater than one for $s = z$, if z is small this limits the band where the sensitivity an be low. Equally the module of W_2 cannot be greater than one for $s = p$, if p is large this may inhibit robustness in the presence of some type of uncertainty.
Situation is still worse when zero and pole are jointly present in the right half plane, especially when one is near the other. A few definitions and properties needed for the following treatment are contained at the end of the section.
With an unstable open loop pole the sensitivity is not minimum phase. It can be factorized as a minimum phase term $S_{fm}(s)$ and an all pass filter

$$S(s) = S_{fm}(s) \cdot \frac{s-p}{s+p}$$

Moreover the presence of a right half plane zero impose:

$$1 = \left| S(z) \right| = \left| S_{fm}(z) \right| \cdot \left| \frac{z-p}{z+p} \right|$$
$$\left| S_{fm}(z) \right| = \left| \frac{z+p}{z-p} \right|$$

Similar considerations apply to the unitary closed loop function with respect to the zero:

$$1 = \left| G_{eq1}(p) \right| = \left| G_{eq1\,fm}(p) \right| \cdot \left| \frac{p-z}{p+z} \right|$$
$$\left| G_{eq1\,fm}(p) \right| = \left| \frac{p+z}{p-z} \right|$$

In conclusion, as only minimum phase terms contribute to the modules, we obtain:

$$1 > \left\| W_1 \cdot S \right\|_\infty = \left\| W_1 \cdot S_{fm} \right\|_\infty \geq \left| W_1(z) \frac{z+p}{z-p} \right|$$
$$1 > \left\| W_2 \cdot G_{eq1} \right\|_\infty = \left\| W_2 \cdot G_{eq1\,fm} \right\|_\infty \geq \left| W_2(p) \frac{p+z}{p-z} \right|$$

The weighting function results greatly penalized, especially when zero and pole are near.

Zeroes and poles in the right half plane

Analysis of singularities of a transfer function on the right half plane requires a few definitions.

Definition 6.3 Minimum phase system

The transfer function of a stable system without zeros in the right half plane has at each frequency a minimum phase with respect to any other transfer function having the same module.

Definition 6.4 All pass filter

A stable dynamical system which has unitary module at all frequencies is called all pass filter. Its transfer function must necessarily have pairs of zeros-poles at the same frequency symmetric with respect to the imaginary axis:

$$G_{pt}(s) = \frac{s-z}{s+z}$$

Definition 6.5 Non-minimum phase system

Any non-minimal phase system can be factorized in the product of a minimum phase system and an all pass filter.

$$G(s) = G_{fm}(s) \cdot G_{pt}(s)$$

6.5 Shaping the loop transfer function

One degree of freedom design is limited to the analysis of the loop transfer function, and the unitary closed loop function is adopted as reference-output relationship. Techniques presented here are taken from [8].
Requirements on sensitivity, dynamic response, uncertainty and control activity originate the weighting functions W_1, W_2 and $\overline{W_3}$ that characterize the performance indices discussed before that we rewrite in the following forms:

$$\begin{aligned} &\Gamma_1 = |W_1 \cdot S| + |W_2 \cdot G_{eq1}| < 1, \quad \Gamma_2 = (|\overline{W_3}| + |W_2|) \cdot G_{eq1} < 1 \; \forall w \\ &\Gamma_1 = \left| W_1 \frac{1}{1+G_a} \right| + \left| W_2 \frac{G_a}{1+G_a} \right| < 1, \quad \Gamma_2 = (|\overline{W_3}| + |W_2|) \cdot \left| \frac{G_a}{1+G_a} \right| < 1 \; \forall w \end{aligned} \tag{6.17}$$

Objective of the design is to shape, through a compensation filter in cascade to the plant, the loop function in order to satisfy (6.17). As usual we subdivide the frequency axis in three bands low frequencies B.F., intermediary frequencies M.F. in the neighbor of

w_c and high frequencies A.F. From (6.17) the following approximate inequalities for $|G_a|$ can be found:

$$|G_a| >> 1: \quad \frac{|W_1|}{|G_a|} + |W_2| < 1, \quad |G_a| > \frac{|W_1|}{1-|W_2|}, \quad |\overline{W}_3| + |W_2| < 1$$

$$|G_a| << 1: \quad |W_1| + |W_2| \cdot |G_a| < 1, \quad |G_a| < \frac{1-|W_1|}{|W_2|}, \quad |G_a| < \frac{1}{|\overline{W}_3| + |W_2|}$$

These conditions clearly show the antagonistic role between performance represented by W_1 and robustness/control activity represented by W_2 and $\overline{W}_3$.

At low frequencies the module of the loop function must stay above the mask defined from performances, at high frequencies it must stay below the mask defined from uncertainty and control bounds.

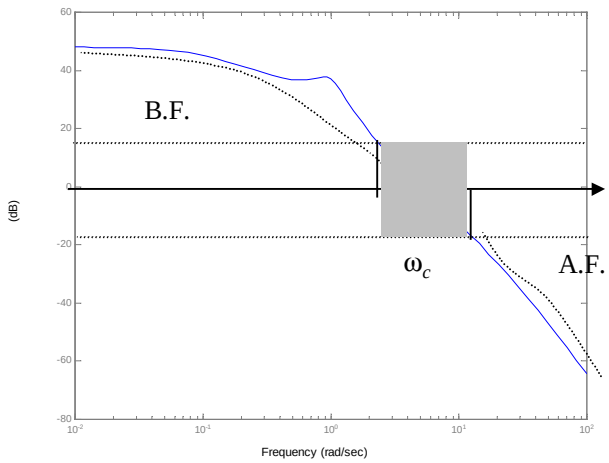


Fig. 6.27 Constraints on the loop function

Part of the conflicts find in the design are solved directly during the choice of the weighting functions. This, obviously, requires a compromise between the requirements. Satisfy performances at low frequencies, especially steady state tracking or regulation errors, is not difficult: it is necessary to operate on the steady state loop gain, and eventually introduce in the compensator one or two pole at the origin. More difficult, eventually, is try to extend the frequency band. Limitations will be evident observing the constraints pose by robustness.

At higher frequencies, to guarantee robustness it is required, once the feedback has completed its action, to drive the loop gain to zero as faster as possible.

The real problem is represented by the intermediary frequency band, where the needs of performance and robustness interfere. In this range the bounds on the module of the loop function discussed so far cannot be used any more. To satisfy robust performance here it is needed:

1. to guarantee a satisfactory phase margin at the frequency w_c;
2. to guarantee a satisfactory gain margin, with fast decay of the gain at the frequencies greater than w_c.

The next section outlines the classical filters proposed in the literature to satisfy previous requirements.

6.6 One degree of freedom design

The classical approach to feedback control is based on a few elemental filters to be used alone or in group to shape the loop transfer function.
Maintaining the subdivision in the three frequency ranges discussed before the compensation filters can classified as:

1. Filters at low frequencies, with the objective to guarantee high precision and hence high steady state gains. These filters operate essentially on the module of the loop function. The are called *lag compensation*;
2. Filters at intermediary frequency with the function to guarantee phase margins. These filters act essentially on the phase of the loop function and are called *lead compensation*;
3. Filters at high frequency with the scope to satisfy gain margins, and hence to reduce the loop gain at higher frequencies. We call them *guardian filters*. A well known example is the *notch filter*

6.6.1 Lag compensation

The first and simplest need of a control loop is to satisfy steady state error conditions. This is achieved assigning to the controller a proper steady state gain, adding if needed for asymptotic conditions one or two poles at the origins. As a consequence a first setting of the module of the loop function results, with the positioning of w_c on the frequency axis.
w_c will determine the unitary closed loop function band (a simple rule states $w_B \approx 2w_c$) and the region of insensitivity of the control loop.
We know that the position w_c is constrained by the masks in frequency resulting from robust performance specifications and the value resulting from the loop gain assignment may not be compatible.
To separate gain needs at low frequencies and robustness requirements at the other frequencies a compensation filter can be introduced, composed of a pair pole-zero positioned at low frequencies, with the pole at the lowest frequency. This is called lag compensation:

$$G_{ci}(s) = \frac{s + m_i \cdot \lambda_i}{m_i \cdot (s + \lambda_i)}, \quad m_i > 1 .$$

In sampled data systems the digital filter for a lag compensation is obtained from the analog filter directly mapping zero and pole

$$G_{ci}(z) = \frac{(1 - e^{-\lambda_i \cdot \Delta}) \cdot (z - e^{-m_i \cdot \lambda_i \cdot \Delta})}{(1 - e^{-\lambda_i \cdot m_i \cdot \Delta}) \cdot (z - e^{-\lambda_i \cdot \Delta})} \cong \frac{(z - 1 + m_i \cdot \lambda_i \cdot \Delta)}{m_i \cdot (z - 1 + \lambda_i \cdot \Delta)}$$

This filter will match with great accuracy the frequency behavior of the corresponding analog filter.

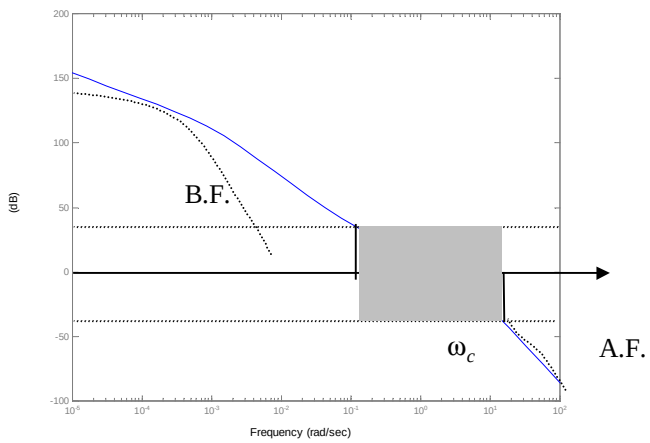


Fig. 6.28 Module of the loop function with lag compensation

It can be seen from figure 6.28 that the compensation filter acts on the module of the loop function.

Phase lag

A side effect of the lag compensation is an undesired reduction of the phase with negative effects on the phase margin. The only design attention is to locate the compensation at sufficiently low frequencies.

Example 6.6 Lag compensation to guarantee error zero in tracking a ramp.

Let be plant given by

$$G(s) = \frac{400}{s\,(s+24)},$$

and the desired closed loop behavior be

$$G_{eqdes}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta = 0.6 \quad \omega_n = 20,$$

with steady state error zero to a reference ramp.
To satisfy the requirements the weighting function W_1 is chosen as

$$W_1(s) = \frac{(s+0.1667)\,(s^2 + 24s + 404.1)}{s^2\,(s+24.17)},$$

let note in particular the presence of two pole at the origin. The weighting function has been chosen following the example 6.5, assigning a G_{eqdes} as in (6.6), in order to satisfy the error condition given by (6.7).
Figure 6.29 shows that the plant without compensation doesn't satisfy low frequency requirements; by adding a lag compensation with the pole at the origin

$$G_c(s) = \frac{s+0.165}{s}.$$

The compensate loop function, coincident with the mask at low frequencies satisfies now the requirements as seen in figure 6.29.

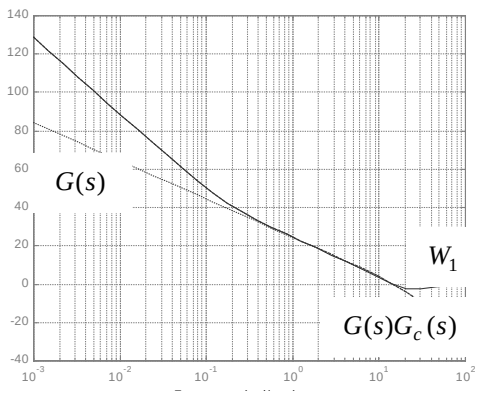


Fig. 6.29 Weighting function and loop function satisfying requirements

Settling time

The presence of a pole at low or very low frequency is not without consequences on the settling time of the response. This is shown by the next observations.
Let consider a first feedback loop of type 1 with a stable loop function given by:

$$G_a = \frac{k^*}{s(s+\lambda)}, \quad k_v = \frac{k^*}{\lambda}$$
$$G_{eq1} = \frac{k^*}{s^2 + \lambda \cdot s + k^*} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \tag{6.18}$$

The tracking error to a unitary ramp, as seen in (6.5), is:

$$e_\infty = \frac{1}{k_v} = \frac{2\zeta}{\omega_n}$$

where the transient response is characterized by the two dominant poles of G_{eq1}.

Let consider now a second example, obtained from the first one by increasing the loop gain of a factor m_i and introducing a lag compensation with a rate zero-pole of the same value.

$$G_a = \frac{k^* \cdot (s + m_i \cdot \lambda_i)}{s(s+\lambda)\cdot(s+\lambda_i)}, k_v = \frac{k^* \cdot m_i}{\lambda}$$
$$G_{eq1} = \frac{k^* \cdot (s + m_i \cdot \lambda_i)}{s(s+\lambda)\cdot(s+\lambda_i) + k^* \cdot (s + m_i \cdot \lambda_i)} \cong \frac{\omega_n^2 \cdot m \cdot (s+z)}{(s^2 + 2\zeta\omega_n + \omega_n^2)\cdot(s + m \cdot z)}$$
$$z = m_i \cdot \lambda_i$$

in order to leave unchanged ω_c and have both closed loop systems with similar dominant dynamics. However, in the second example the lag compensation will add a pole-zero pair. Let say that the resulting rate of the pole-zero pair in the closed loop is m [12]. This rate, from the conditions of tracking error (6.7), will be slightly greater than one:

$$e_\infty = \frac{1}{k_v \cdot m_i} = \frac{1}{k_v} - \frac{m-1}{m \cdot z}$$
$$m = \frac{m_i \cdot k_v}{m_i \cdot k_v - (m_i - 1)\cdot z}$$

The system response in the second case will result from the contribution of the dominant dynamics and the low frequency pole introduced by the lag compensation. An expansion of the step response to simple fractions gives:

[12] Attention we are considering closed loop poles so m is much different than m_i, the rate between zero and pole in open loop

$$y(s)=\frac{\omega_n^2\cdot m\cdot(s+z)}{(s^2+2\zeta\omega_n+\omega_n^2)\cdot(s+m\cdot z)}\frac{1}{s}=\frac{\omega_n^2}{(s^2+2\zeta\omega_n+\omega_n^2)}\frac{1}{s}+\frac{h}{s+m\cdot z}$$

The first term is the dominant dynamics identical to the starting example (6.18). The second term is an exponential with residual h that, considering

$$\left.\frac{\omega_n^2}{(s^2+2\zeta\omega_n+\omega_n^2)}\right|_{s=-m\cdot z}\cong 1,$$

is given by

$$\frac{h}{s+m\cdot z},\quad h=m-1\Leftrightarrow(m-1)\cdot e^{-m\cdot z\cdot t};$$

$$\int_0^\infty(m-1)\cdot e^{-m\cdot z\cdot t}\cdot dt=\frac{m-1}{m\cdot z}.$$

Let note that the value of the integral of the exponential response coincides with the reduction of the steady state tracking error to the ramp introduced by the compensation. Hence, with identical error reduction, m and z can assume different values, but the time constant ($1/mz$) of the pole will be always significant with respect to the dominant response, with consequences on the settling time. This is the cost paid for the introduction of the lag compensation. Moreover, if m-1 is too large, along with settling time, it will influence also the overshoot.

Example 6.7 Settling time with lag compensation

This example approaches the same problem discussed in the previous example. Let be the desired dominant closed loop dynamics given by

$$G_{eqdes}(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},\quad \zeta=0.6\quad \omega_n=20$$

The ramp tracking error is

$$\frac{2\zeta}{\omega_n}=0.06$$

we want to modify the closed loop dynamics in order to bring this error to zero, by introducing a lag compensation. For this we need

$$\frac{m-1}{m\cdot z}=0.06\,.$$

In the selection of the parameters we consider two extreme conditions:

$$\begin{aligned}&m\cdot z=0.83,\quad m-1=0.05\\&m\cdot z=0.17,\quad m-1=0.01\end{aligned}.$$

The resulting exponential contribution to the response affecting the settling time in the tow cases is show in the next figure

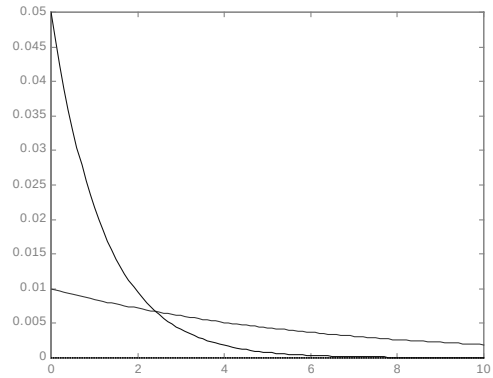


Fig. 6.30 Settling time for two different design choices

The effects on the whole response are next

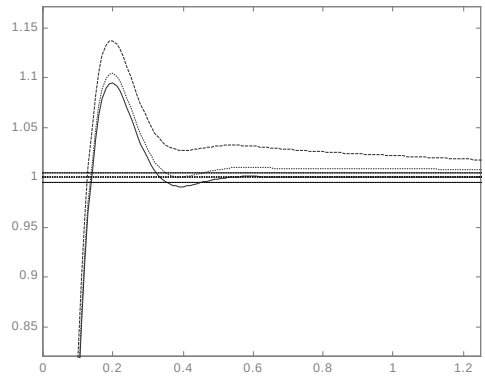


Fig. 6.31 Settling time for a tolerance ± 0.005

6.6.2 Lead compensation

Once low and high frequencies have been considered in a preliminary design, the possible range of frequencies where G_{eq1} and S will likely have a maximum is in the neighbor of w_c, with a peak magnitude dependent on the loop stability margins. A technique

to improve the phase margin is to introduce a zero-pole compensation, with the zero near w_c and the pole at higher frequency, called lead compensation:

$$G_{ca}(s) = \frac{m_a \cdot (s + z_a)}{s + m_a \cdot z_a}, \quad m_a > 1$$

At difference of lag compensation, this filter acts essentially on the loop phase.
In sampled data system the digital lead compensation has a behavior markedly different from the corresponding analog compensation, so the mapping of zero and pole from continuous to discrete cannot be used, and the filter has to be designed directly in the *z* plane using a proper abacus.

Example 6.8 Lead Compensation to guarantee stability margins

Let be the open loop system given by

$$G(s) = \frac{400}{s^2\,(s + 24)},$$

and the performance weighting function

$$W_1^{-1} = \frac{2\,s^2}{s^2 + 5\,s + 6.25}.$$

This function will guarantee a desired frequency band to the closed loop function, as well as a stability margin.
The plant alone, without compensation satisfies requirements at low frequencies however the closed loop system will not be stable.

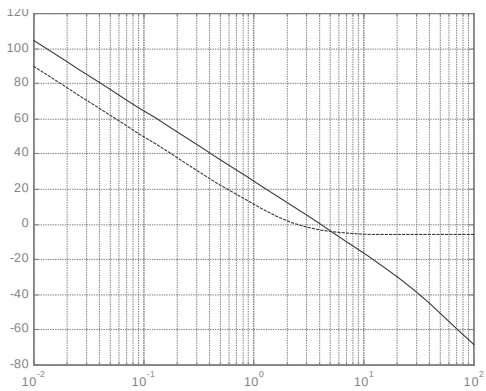


Fig. 6.32 Comparison of loop function and weighting function

We propose two different lead compensations with zero and pole at the left and right, respectively, of frequency $4\,\frac{rad}{s}$.

$$G_{ca1}(s) = \frac{5\,(s+2)}{s+10}$$
$$G_{ca2}(s) = \frac{8\,(s+3)}{s+24}.$$

Both compensations stabilize the control loop, the first, however, doesn't offer the desired stability margin (dashed line on the figure).

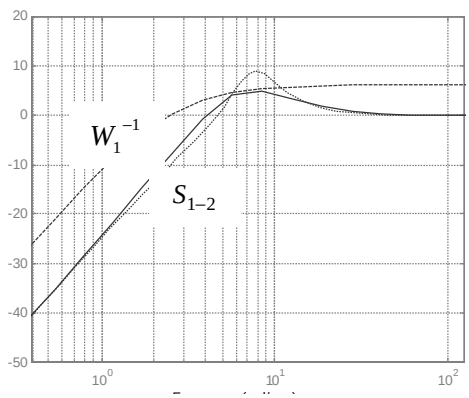


Fig. 6.33 Two possible lead compensations

Disadvantages

The negative aspect of a lead compensation is represented by the increase of gain of a factor of m_a it introduces at high frequencies. This gain increase reduces the gain margin with a negative effect to robustness of the control, and it raises the control activity. To emphasize this last aspect note that the closed loop relationship between measurement noise and control is approximately:

$$G_{equ} = G_c \cdot S \cong G_c, \quad w > w_c \,.$$

Example 6.9 Control activity as a consequence of a lead compensation

The previous example is extended by introducing a further specification on the maximum module of G_{equ}, expressed by the weighting function:

$$W_3 = \frac{1}{8}, \quad \overline{W_3} = \frac{1}{8}\frac{s^2(s+24)}{400}.$$

We verify, that the second of the two compensators proposed at the beginning satisfies the further requirement (continuous line in the figure).

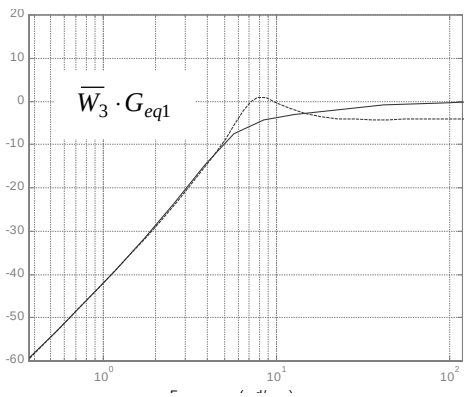


Fig. 6.34 Requirement on the control activity

6.6.3 High frequency filters

As a general rule multiplicative plant uncertainty grows with the frequency and the most critical region for satisfying robust stability is clearly immediately after w_c. Robustness in this range of frequencies is guaranteed by improving the gain margin obtained by introducing low pass or band rejection filters. We call them guardian filters. A typical example of a guardian is represented by "*notch filters*" obtained by pairs of complex conjugate zeros and poles at the same natural frequency but different damping (damping of the zeros is chosen much smaller than the one of the poles):

$$G_n(s) = \frac{s^2 + 2\zeta_z \omega_n s + \omega_n^2}{s^2 + 2\zeta_p \omega_n s + \omega_n^2}$$

Another example of guardian filters is given by pure low pass filter:

$$G_{pb}(s) = \frac{\lambda^b}{(s+\lambda)^b}, \quad b = 1,2,..$$

Compared with a notch filter, pure low pass filters show a significant loss in phase that extends to lower frequencies, with dangers to the phase margin.

Example 6.10 Notch filter to guarantee robust stability

Guardian filters operate at high frequencies, Let apply them to a problem of robust stability.
The process has a nominal model identical to previous example 6.8, However the plant is affected by parasitic dynamics represented by a pair of complex conjugate poles at high frequency with low damping

$$G_p = \frac{400 \cdot 225}{s^2(s+24)(s^2+2s+225)} = \frac{400}{s^2(s+24)}(1+W_2\Delta), \quad W_2 = \frac{s(s+2)}{s^2+2s+225}.$$

Both lead compensation designs of the previous example do not satisfy robust stability in this condition, as shown by the robustness index $|W_2 G_{eq1}|$ which crosses the 0 dB line.

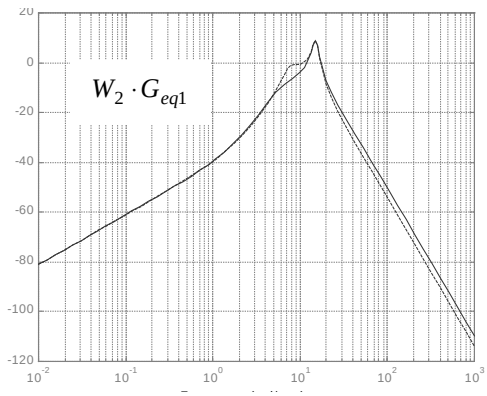


Fig. 6.35 Failure of robust stability condition for two alternative compensations

We add a notch filter to the original compensation to improve robustness

$$G_{notch} = \frac{s^2+225}{s^2+15s+225}.$$

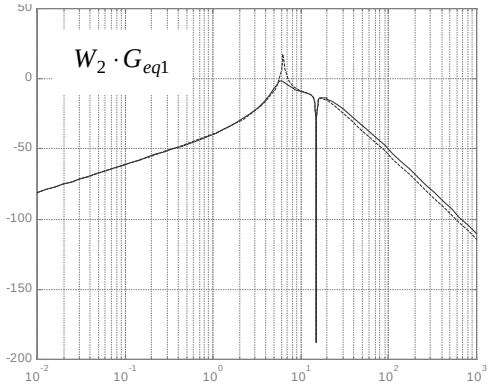


Fig. 6.36 Effect of a notch filter in robust stability

With the notch filter the second of the two compensations guarantees robust stability.

Disadvantages

The reduction of gain proper of a guardian filter is accompanied by a reduction of phase that propagates toward the lower frequencies endangering the loop phase margin.

6.7 Two degrees of freedom design

The reason for adopting a two degree of freedom design is the conflict between requirements of the reference-output response and the sensitivity of the closed loop. We know that with one degree of freedom design unitary closed loop function and sensitivity cannot be selected independently. Figure 6.37 shows clearly the situation: in the Nyquist plot the frequency where the loop function enters in the unitary circle with center the critical point causing the sensitivity module to become more than one is in general smaller than the frequency where the loop function crosses the M circle of value 0.707 indicating the G_{eq1} band w_B. Hence the closed loop function G_{eq1} cannot be insensitivity to uncertainty inside its whole frequency band.
To achieve a greater freedom in the design a two degree of freedom control structure is needed.

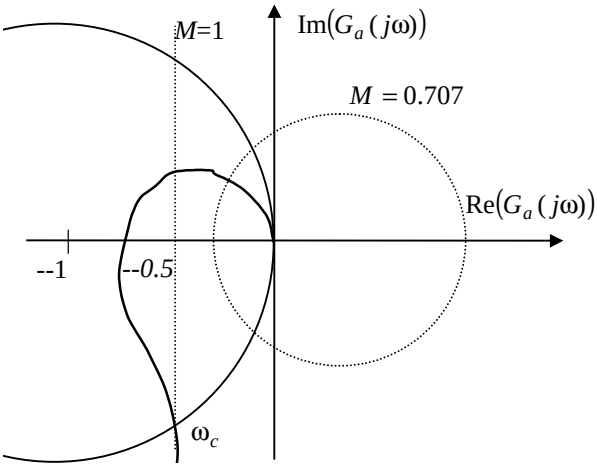


Fig. 6.37 Comparison in frequency of $G_{eq1}(j\omega)$ and $S(j\omega)$

The objective is to relate the reference-output closed loop function with the sensitivity as shown in the next figure.

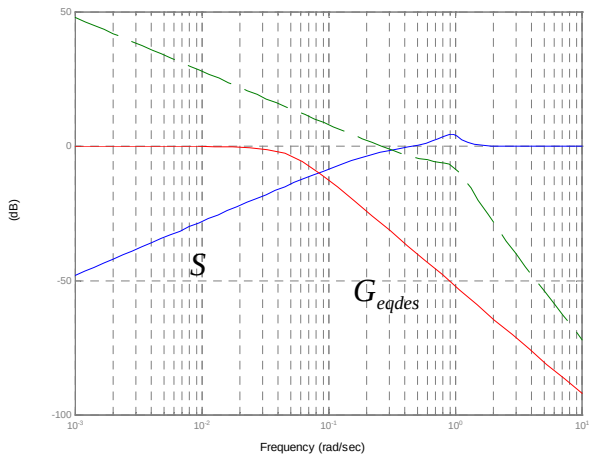


Fig. 6.38 $G_{eqdes}(j\omega)$ with insensivity extended up to the limits of its band

In chapter 6.8 several equivalent structures of two degree of freedom controllers were presented characterized by two compensation blocks, for simplicity we consider here the structure characterized by one block in cascade and one block in feedback:

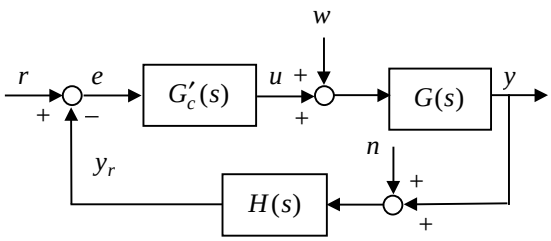


Fig. 6.39 Two degree of freedom control – first form

Independently from the input-output characteristics, the selection of the loop function, the sensitivity to disturbance the robustness of the control are problems approached identically in both one degree and two degree of freedom design. So, we hypothesize that the loop function G_a has already been designed.

We call G_{eqdes} the desired reference-output closed loop function. This function has been chosen to satisfy the requirement of tracking, both during the transient and at steady state.

The two degree of freedom design starts from these two initial elements, exploiting algebraic considerations applied to the control structure of figure 6.39.

$$G_{eqdes}(s) = \frac{G_c'(s) \cdot G(s)}{1 + G_a(s)}; \quad G_a(s) = G_c(s) \cdot G(s)$$

$$G_c(s) = G_c'(s) \cdot H(s)$$

$$G_{eqdes}(s) = \frac{G_a(s)}{1 + G_a(s)} \cdot \frac{1}{H(s)}$$

$$G_c'(s) = \frac{G_{eqdes}(s) \cdot [1 + G_a(s)]}{G(s)} = \frac{n_{eqdes}(s) \cdot d_{cc}(s) \cdot d(s)}{d_{eqdes}(s) \cdot d_a(s) \cdot n(s)} = \frac{n_{eqdes}(s) \cdot d_{cc}(s)}{d_{eqdes}(s) \cdot d_c(s) \cdot n(s)}$$

$$\dim(G_c') = \frac{\dim(n_{eqdes}) + \dim(d_a)}{\dim(d_{eqdes}) + \dim(d_c) + \dim(n)}$$

$$H(s) = \frac{G_c(s)}{G_c'(s)} = \frac{G_a(s)}{[1 + G_a(s)] \cdot G_{eq_d}(s)} = \frac{n_a(s) \cdot d_{eqdes}(s)}{n_{eqdes}(s) \cdot d_{cc}(s)}$$

$$\dim(H) = \frac{\dim(n_a) + \dim(d_{eqdes})}{\dim(n_{eqdes}) + \dim(d_{cc})}$$

Obviously, the two control filters must be physically realizable and they must preserve the internal stability of the loop.

Physical realizability of the filters

A completely arbitrary G_{eqdes} doesn't assure that the resulting compensation filters will be proper. For this it may be necessary to add some more poles to G_{eqdes}. These poles will be selected outside the desired frequency band to not modify significantly the desired dominant behavior: We indicate the new function:

$$G_{eq}(s) = G_{eqdes}(s) \cdot \frac{1}{\prod (1 + s / \tilde{\lambda}_i)}$$

The number of these poles must guarantee the properness of the control filters

$$G_c'(s) = \frac{n_{eqdes}(s) \cdot d_{cc}(s)}{d_{eqdes}(s) \cdot \prod (1 + s / \tilde{\lambda}_i) \cdot d_c(s) \cdot n(s)}$$

$$H(s) = \frac{n_a(s) \cdot d_{eqdes}(s) \cdot \prod (1 + s / \tilde{\lambda}_i)}{n_{eqdes}(s) \cdot d_{cc}(s)}$$

Simplifications by cancellation of pairs zero-pole

When the control filters result unnecessarily complex for the presence of pair zero-pole close each other, these pairs can be cancelled from the filters, they appear, however, in the closed loop function. Let indicate with $\tilde{n}$ and $\tilde{d}$ the polynomials containing respectively the poles and zeros to cancel. The compensation filters and the closed loop functions are modified in

$$G_c'(s) = \frac{n_{eqdes}(s) \cdot \tilde{n}(s) \cdot d_{cc}(s)}{d_{eqdes}(s) \cdot \tilde{d}(s) \cdot d_c(s) \cdot n(s)}$$

$$G_{eq}(s) = G_{eqdes}(s) \cdot \frac{\tilde{n}(s)}{\tilde{d}(s)}; \quad \lim_{s \to 0} \frac{\tilde{n}(s)}{\tilde{d}(s)} = 1$$

Decision on possible simplifications must consider the consequences in the closed loop response.

Plant zeros in the right half plane

To guarantee internal stability no perfect cancellations between zeros and poles in the right half plane must exist.
If the plant has one zero in the right half plane, and we do not include this zero in the desired reference-output closed loop function a pole in the same position of the zero will appear in one of the two control filters. To avoid this situation, the right half plane zero of the plant must be maintained in the G_{eq}, adding eventually a pole with the same module on the left half plane to guarantee the physical realizability of the filters.

To achieve this result we must modify G_{eqdes} introducing the pair zero on the right half plane and pole on the left half plane. Let be $\tilde{\lambda}$ and $\tilde{\tilde{\lambda}}$ the values of this pair, $G_c'(s)$ modifies as follows:

$$G_c'(s) = \frac{n_{eqdes}(s) \cdot (1 - s/\tilde{\lambda}) \cdot d_{cc}(s)}{d_{eqdes}(s) \cdot (1 + s/\tilde{\tilde{\lambda}}) \cdot d_c(s) \cdot n(s)}$$

where the factor $(1 - s/\tilde{\lambda})$ simplifies with the corresponding factor present in $n(s)$. The closed loop function is different than desired

$$G_{eq}(s) = G_{eqdes}(s) \cdot \frac{(1 - s/\tilde{\lambda})}{(1 + s/\tilde{\tilde{\lambda}})}$$

but this is unavoidable.

Example 6.11 Requirement on the sensitivity of the desired reference-output response

The system is identical of the one in example 6.8

$$G(s) = \frac{400}{s^2\,(s + 24)},$$

with a sensitivity guaranteed from the weighting function

$$W_1^{-1} = \frac{2\,s^2}{s^2 + 5\,s + 6.25}.$$

Let add the requirement on the desired closed loop function

$$G_{eqdes} = \frac{0.25}{s^2 + 0.6\,s + 0.25},$$

It is easy to verify that the chosen G_{eqdes} enjoys a certain level of insensitive to uncertainty for its whole frequency band.

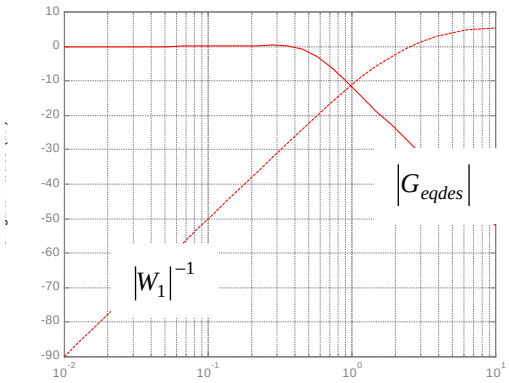


Fig. 6.40 Insensitivity of the closed loop function

The filter that compensates the loop is chosen as

$$G_{ca2}(s) = \frac{8\,(s+3)}{s+24}.$$

The characteristic polynomial of the closed loop with this compensation is

$$d_{cc}(s) = (s+33.35)(s+8.316)(s^2+6.339s+34.62).$$

The filters for the two degree of freedom structure are

$$G_c'(s) = \frac{6.25\cdot 10^{-4}(s+33.35)(s+8.316)(s^2+6.339s+34.62)}{(s^2+0.6\,s+0.25)\cdot(1+s/\tilde{\lambda}_i)\cdot(s+24)}$$

$$H(s) = \frac{12800(s+3)\cdot(s^2+0.6\,s+0.25)\cdot(1+s/\tilde{\lambda}_i)}{(s+33.35)(s+8.316)(s^2+6.339s+34.62)}$$

For physical realizability one more pole is introduced in the closed loop function, we select $\tilde{\lambda}_i = 5$.

To test performances we double the plant steady state gain. In the next figure we show the step response of the unitary gain closed loop function G_{eq1} (resulting from one degree of freedom design) in the nominal and perturbed cases. A great sensitivity to uncertainty is clearly evident in the two responses.

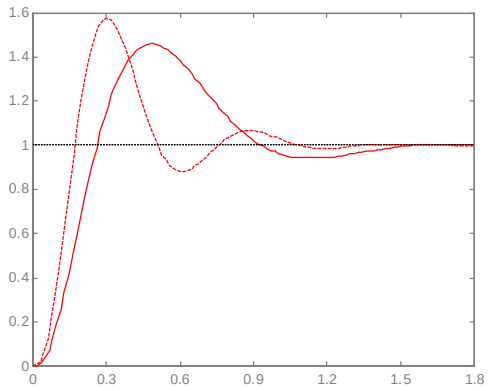


Fig. 6.41 Sensitivy of the response in one degree of freedom design

The next figure, viceversa, shows perturbed and nominal responses of G_{eqdes}. Differences between the two responses are negligible.

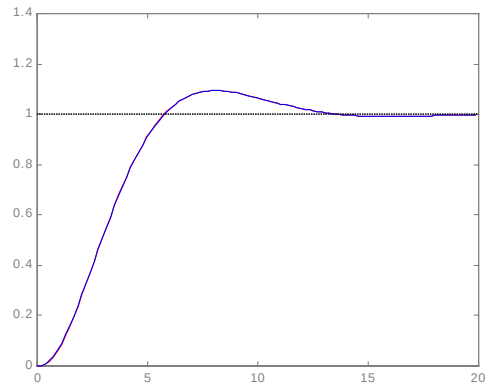


Fig. 6.42 Insensitivity of the response in a two degree of freedom design

The time response is slower in this case than in the previous one, but the reduction of frequency band with respect to G_{eq1} is a necessity to improve insensitivity to uncertainties.

6.8 Partial state feedback

In this section we show how classical techniques approach a class of closed loop controls with more than one feedback loop.

The special case of interest is when output and its derivative are measured contemporaneously

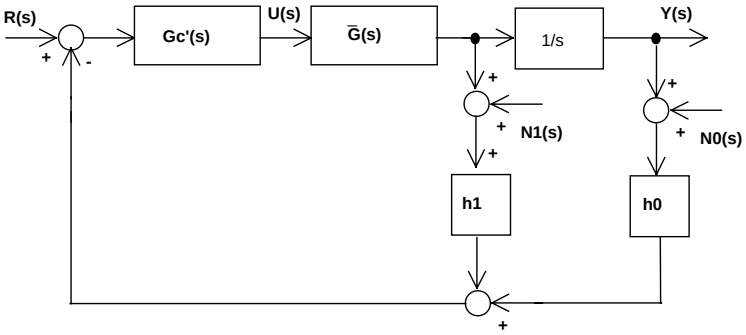


Fig. 6.43 Partial state feedback – output and its derivative

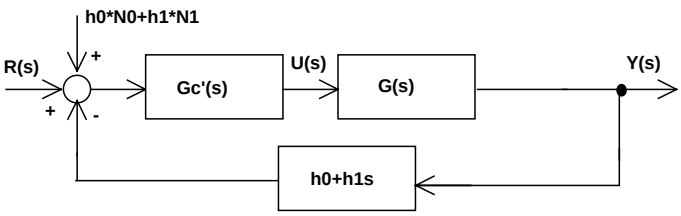


Fig. 6.44 Feedback loop in the presence of partial state feedback

Let compensate the plant $G(s)$ using a unitary feedback on the output with a cascade filter $G_c(s)$. Using a partial state feedback we can replicate the same loop function with a new cascade filter $G_c'(s)$ that differs from $G_c(s)$ for the dynamics introduced by the feedback.

The closed loop dynamics will be identical in the two cases, however the relationships between signals in the loops and the sensitivity function will be different. The design can exploit the added freedom offered by the partial state feedback to improve the activity of the control generated by measurement noises.

Let consider a case of lead compensation of the control loop, with a measurement noise at high frequency $n_0(s)$. The control activity is given by:

$$u(s) = \frac{k_c \cdot m_a \cdot (s + z_a)}{(s + m_a \cdot z_a)} S(s) \cdot n_0(s)$$

Using a partial state feedback measuring the output derivative we can substitute the zero of the lead compensation. Guaranteeing the unitary gain of the feedback we have:

$$h_0 = 1, \quad h_1 = \frac{1}{z_a}, \quad G_c^{'}(s) = \frac{k_c \cdot m_a \cdot z_a}{(s + m_a \cdot z_a)}$$

$$u(s) = \frac{k_c \cdot m_a \cdot z_a}{(s + m_a \cdot z_a)} S(s) \cdot (n_0(s) + \frac{1}{z_a} \cdot n_1(s))$$

In the first case the unique output measurement noise is amplified at high frequency of a gain $k_c \cdot m_a$ introduced by the lead compensation. In the second case we have two disturbances that, however, are filtered by the pole in the cascade compensation. Usually, in spite of the presence of two disturbances, the control activity in this second case is reduced.

Let assume that disturbances are white noises with respective variances σ_0^2, σ_1^2, we can adopt as performance of the control activity the H_∞ norm of the root square of the input power spectra weighted by a performance measure W_3, in the two cases we have:

$$\left\| W_3 \frac{k_c \cdot m_a \cdot (s + z_a)}{(s + m_a \cdot z_a)} S(s) \cdot \sigma_0 \right\|_\infty < 1$$

$$\left\| W_3 \frac{k_c \cdot m_a \cdot z_a}{(s + m_a \cdot z_a)} S(s) \cdot \sqrt{\sigma_0^2 + \frac{1}{z_a^2} \cdot \sigma_1^2} \right\|_\infty < 1$$

Transferring previous conditions to one of the standard forms of robust performance, remembering that

$$G_a(s) = \frac{k_c \cdot m_a \cdot (s + z_a)}{(s + m_a \cdot z_a)} G(s),$$

these can be written in a common form

$$\left\| \frac{W_3}{G(s)} \sigma_0 \cdot G_{eq1}(s) \right\|_\infty < 1$$

$$\left\| \frac{W_3}{G(s) \cdot (1 + \frac{s}{z_a})} \sqrt{\sigma_0^2 + \frac{1}{z_a^2} \cdot \sigma_1^2} \cdot G_{eq1}(s) \right\|_\infty < 1$$

The interpretation of the partial state feedback can be better understood comparing the two weighting functions resulting from the two cases:

$$\overline{W_3} = \frac{W_3}{G(s)} \cdot \sigma_0 \quad \text{without partial feedback}$$
$$\overline{W_3} = \frac{W_3}{G(s) \cdot (1 + \frac{s}{z_a})} \sqrt{\sigma_0^2 + \frac{1}{z_a^2} \cdot \sigma_1^2} \quad \text{with partial feedback} \tag{6.19}$$

In the previous results we have exactly replicated the zero of the cascade compensation with the partial state feedback. In reality the zero introduced by the feedback can assume any value, with the condition to maintain unchanged the loop function, by adding an identical pole in the compensation. The new weighting function in this condition is:

$$\overline{W_3} = \frac{W_3}{G(s) \cdot (1 + h_1 \cdot s)} \sqrt{\sigma_0^2 + h_1^2 \cdot \sigma_1^2}$$

Example 6.12 Requirements on the control activity achieved using partial state feedback

The system is the one of example 6.8

$$G(s) = \frac{400}{s^2\,(s+24)},$$

with similar performance requirements, and a white measurement noise of variance

$$\sigma_0^2 = 10^{-6}.$$

Let impose that the power spectra of the control activity due to the noise satisfies

$$|S_u| = |G_{equ}|^2 \cdot \sigma_0^2 < 10^{-6} \ \forall w.$$

This is equivalent to choose in (6.19) a weighting function

$$W_3 \cdot \sigma_0 = 1, \quad W_3 = 10^3$$

We can seen that this requirement cannot be satisfied by the two lead compensations proposed in the example. Adopt the lead compensation

$$G_{ca2}(s) = \frac{8\,(s+3)}{s+24}$$

and introduce a partial state feedback. Both measures have an independent white noise with identical variance. With reference to figure 6.44 we select $h_1 = 1$, the cascade compensation filter modifies in

$$G_{ca2}'(s) = \frac{8(s+3)}{(s+24)(s+1)}$$

with a feedback block

$$h_0 + h_1 \cdot s = 1 + s\,.$$

The resulting weighting function is

$$\frac{W_3}{(1+s)}\sigma_0\sqrt{2} = \frac{1.41}{(1+s)}\,.$$

The function $\overline{W}_3$ is determined consequently. In the next figure we can see the module of $\overline{W}_3$ without and with partial state feedback, and in the figure 6.46 we see that $\left|G_{eqdes}\overline{W}_3\right|$ satisfies control activity bounds only in the presence of partial state feedback.

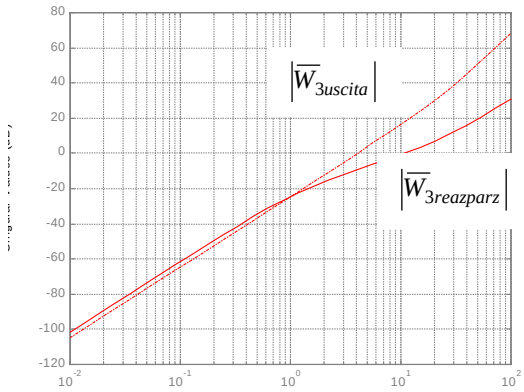


Fig. 6.45 Weighting functions with and without partial state feedback

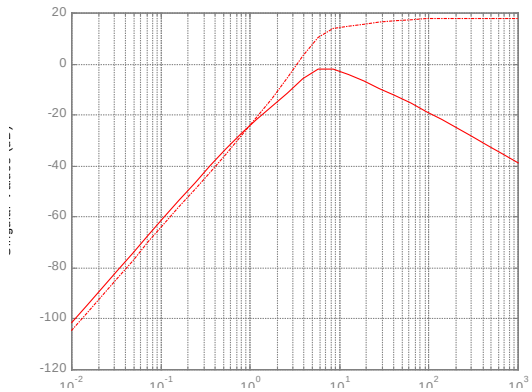


Fig. 6.46 Performance indices with and without partial state feedback

6.9 Bibliography

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